On p-homogeneous systems of differential equations and their linear perturbations

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Abstract

Long-time asymptotic properties of the solutions of the system

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}),$$

where $\mathbf{f}(\mathbf{u})$ is positive homogeneous of degree $p > 1$, are studied. We also consider the corresponding linearly perturbed system

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}) + A\mathbf{u}.$$

It is shown that if $A = \alpha I$, then the global existence of all solutions for one value of α implies that the same property holds for all α, and that all solutions converge to the origin when $\alpha < 0$. On the other hand, it is shown that the addition of a matrix for which $A\mathbf{u}$ is inward in the sense that $\mathbf{u} \cdot A\mathbf{u} < 0$ for $\mathbf{u} \neq \mathbf{0}$ can turn a p-homogeneous system all of whose solutions are bounded into one which has solutions which blow up in a finite time.

Keywords. growup, blowup, linear perturbation, inward perturbation.

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1 Introduction

We shall say that the d-dimensional system of ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}) \tag{1.1}$$

is p-homogeneous if the d components of $\mathbf{f}$ are positive homogeneous functions of degree p of the d-vector $\mathbf{u}$. That is, $\mathbf{f}(\lambda\mathbf{u}) = \lambda^p\mathbf{f}(\mathbf{u})$ when $\lambda \geq 0$. In this paper we shall study the qualitative features of the solution set of such a system with $p > 1$, and of the systems obtained by adding a linear function of $\mathbf{u}$ to the right-hand side.

Some properties of general p-homogeneous equations will be stated and proved in Section 2. In particular, sufficient conditions for the origin to be a global attractor, as well as for the system to have solutions which blow up in a finite time will be given. More complete results can be obtained in the special case when the number of components d is two, and these results will be derived in Section 3. Sections 4 to 6 are devoted to studying systems whose right-hand side is the sum of a p-homogeneous function and a linear function. Theorem 4.1 gives a rather complete result when the linear term is a multiple of $\mathbf{u}$. The case of a general linear function turns out to be considerably more difficult, and we only obtain some partial results. In particular, Proposition 4.2 shows that the condition that the radial component of the linear vector field is uniformly inward is not sufficient to preserve the property that the origin is a global attractor. Section 5 again deals with the case of two components. Here we find a sufficient condition for the linear perturbation to preserve the property that the origin is a global attractor for the p-homogeneous equation (1.1). Theorem 6.1 of Section 6 provides a tool for proving that linear terms added to the right-hand side of a two-dimensional p-homogeneous system can produce solutions which blow up. Example 6.1 uses this result to obtain a p-homogeneous system all of whose solutions are bounded such that an arbitrarily small linear perturbation produces solutions which blow up in a finite time. This is a simpler example of a phenomenon found in [2]. Example 6.2 gives a finite-difference analog of an example of Mizoguchi, Ninomiya, and Yanagida [1] which shows that the introduction of unequal diffusions in a continuous-space reaction-diffusion system can produce blowup.

2 On solutions of p-homogeneous equations

Throughout this work we shall assume that the vector-valued function $\mathbf{f}(\mathbf{u})$ in (1.1) is continuously differentiable and positive homogeneous of degree $p > 1$.

We observe that if one defines the polar coordinates

$$\begin{aligned} r &= |\mathbf{u}| \\ \boldsymbol{\omega} &= r^{-1}\mathbf{u}, \end{aligned} \tag{2.1}$$

and uses the homogeneity of $\mathbf{f}$, the system (1.1) becomes

$$\begin{aligned} \frac{d\boldsymbol{\omega}}{dt} &= r^{p-1}\mathbf{h}(\boldsymbol{\omega}) \\ \frac{d\ln r}{dt} &= r^{p-1}\ell(\boldsymbol{\omega}), \end{aligned}$$

where

$$\ell(\boldsymbol{\omega}) := \boldsymbol{\omega}\cdot\mathbf{f}(\boldsymbol{\omega}) \text{ and } \mathbf{h}(\boldsymbol{\omega}) := \mathbf{f}(\boldsymbol{\omega}) - \ell(\boldsymbol{\omega})\boldsymbol{\omega}. \tag{2.2}$$

If we introduce the artificial time variable $\tau = \int r^{p-1}dt$, we obtain the equivalent system

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{h}(\boldsymbol{\omega}) \tag{2.3}$$

$$\frac{d\ln r}{d\tau} = \ell(\boldsymbol{\omega}) \tag{2.4}$$

$$\frac{dt}{d\tau} = r^{1-p}, \tag{2.5}$$

in which the equation (2.3) represents a flow on S^{d-1} which is uncoupled from the other equations. Its orbits are the radial projections of the orbits of (1.1) to the unit sphere. Once one has a solution of this system, one can use quadrature to solve the equation (2.4) for $\ln r$ and then (2.5) for t. This gives a parametric representation of the solutions of (1.1).

The following scaling result will sometimes be used:

Lemma 2.1 *If* $\mathbf{u}(t)$ *is a solution of* (1.1) *and* ρ *is any positive constant, then the function* $\mathbf{v}(t) := \rho\mathbf{u}(\rho^{p-1}t)$ *is also a solution of* (1.1).

Proof. The chain rule and the p-homogeneity of $\mathbf{f}$ show that

$$\mathbf{v}_t = \rho^p\mathbf{u}_t = \rho^p\mathbf{f}(\mathbf{u}(\rho^{p-1}t)) = \mathbf{f}(\mathbf{v}). \qquad \square$$

Since $\mathbf{v}/|\mathbf{v}| = \mathbf{u}/|\mathbf{u}|$ and $|\mathbf{v}| = \rho|\mathbf{u}|$, the phase space of (1.1) is invariant under positive dilation, and all dilates of an orbit have the same radial projection on S^{d-1}.

The above decomposition permits us to study the p-homogeneous system (1.1) in terms of the solutions of the system (2.3). As with any system, one first looks at the equilibrium points of (2.3). It is obvious from (2.2) that an equilibrium point $\boldsymbol{\omega}^*$ of this system is characterized by the fact that

$$\mathbf{f}(\boldsymbol{\omega}^*) = \ell(\boldsymbol{\omega}^*)\boldsymbol{\omega}^*. \tag{2.6}$$

It is useful to know the behavior of solutions whose radial projections are equal to or approach an equilibrium point $\boldsymbol{\omega}^*$.

Theorem 2.1 *Let $\mathbf{u}(t)$ be a solution of (1.1) whose radial projection $\boldsymbol{\omega} = \mathbf{u}/|\mathbf{u}|$ is either equal to or converges as τ goes to infinity to an equilibrium point $\boldsymbol{\omega}^*$ which satisfies (2.6). Then*

a. if $\ell(\boldsymbol{\omega}^*) > 0$, $\mathbf{u}$ blows up in a finite time;

b. if $\ell(\boldsymbol{\omega}^*) < 0$, $\mathbf{u}$ approaches the origin as $t \to \infty$;

c. if $\ell(\boldsymbol{\omega}^*) = 0$ so that $\mathbf{f}(\boldsymbol{\omega}^*) = \mathbf{0}$, then all points of the line $\mathbf{u} = r\boldsymbol{\omega}^*$ are equilibria of (1.1), but a solution in which $\boldsymbol{\omega}(t) \neq \boldsymbol{\omega}^*$ can have any asymptotic behavior, depending on the behavior of the function $\ell(\boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \mathbf{f}(\boldsymbol{\omega})$ near the point $\boldsymbol{\omega}^*$.

Proof. We see from (2.3) that if the initial value of $\boldsymbol{\omega}$ is $\boldsymbol{\omega}^*$, then $\boldsymbol{\omega} = \boldsymbol{\omega}^*$ for all t. Thus the solution takes the form $\mathbf{u} = r(t)\boldsymbol{\omega}^*$, where

$$\begin{aligned} r_t &= \ell(\boldsymbol{\omega}^*) r^p \\ r(0) &= r_0. \end{aligned}$$

Since $p > 1$, r blows up in a finite time if $\ell(\boldsymbol{\omega}^*) > 0$, goes to zero if $\ell(\boldsymbol{\omega}^*) < 0$, and remains constant if $\ell(\boldsymbol{\omega}^*) = 0$. This proves the Theorem when $\boldsymbol{\omega}_0 = \boldsymbol{\omega}^*$.

When $\boldsymbol{\omega}_0 \neq \boldsymbol{\omega}^*$, we see from (2.4) and (2.5) that

$$\frac{dr^{1-p}}{dt} = -(p-1)\ell(\boldsymbol{\omega}(\tau(t))). \tag{2.7}$$

By hypothesis, $\ell(\boldsymbol{\omega}(\tau(t)))$ converges to $\ell(\boldsymbol{\omega}^*)$. Thus if $\ell(\boldsymbol{\omega}^*) > 0$, there is a t_1 such that the right-hand side is bounded above by $-(p-1)\ell(\boldsymbol{\omega}^*)/2$ when $t \geq t_1$. Therefore

$$r(t)^{1-p} \leq r(t_1)^{1-p} - \frac{(p-1)\ell(\boldsymbol{\omega}^*)}{2}(t - t_1) \tag{2.8}$$

for $t \geq t_1$. Since $p > 1$, r must become infinite at a finite value of t.

If $\ell(\boldsymbol{\omega}^*) < 0$, the right-hand side of (2.7) is bounded below by $-(p-1)\ell(\boldsymbol{\omega}^*)/2$ for $t \geq t_1$, and one obtains (2.8) with $\leq$ replaced by $\geq$. Since $\ell(\boldsymbol{\omega}^*) < 0$, this shows that r approaches zero as t goes to infinity.

In order to treat the case $\ell(\boldsymbol{\omega}^*) = 0$, we observe that the right-hand sides of (2.3) and (2.4) are independent of each other in the following sense: Choose any smooth scalar function $\ell(\boldsymbol{\omega})$ and any smooth tangent vector field $\mathbf{h}(\boldsymbol{\omega})$ on S^{d-1}. Then if one defines the p-homogeneous vector field

$$\mathbf{f}(\mathbf{u}) := |\mathbf{u}|^p \left[\mathbf{h}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right) + \ell\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\frac{\mathbf{u}}{|\mathbf{u}|} \right], \tag{2.9}$$

the functions ℓ and $\mathbf{h}$ are given by (2.2).

Suppose then that we have a solution $\hat{\boldsymbol{\omega}}(\tau) \in S^{d-1}$ of the system

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{h}(\boldsymbol{\omega})$$

with $\boldsymbol{\omega} \cdot \mathbf{h}(\boldsymbol{\omega}) \equiv 0$ such that $\lim_{\tau\to\infty} \hat{\boldsymbol{\omega}}(\tau) = \boldsymbol{\omega}^*$. Then $\mathbf{h}(\boldsymbol{\omega}^*) = \mathbf{0}$. The mapping $\tau \to \hat{\boldsymbol{\omega}}(\tau)$ is one-to-one. Choose any smooth positive function $\hat{r}(\tau)$ such that $\hat{r}'(\tau)/\hat{r}(\tau)$ converges to zero as $\tau \to \infty$. Define a function $\ell(\boldsymbol{\omega})$ on S^{d-1} by setting

$$\ell(\hat{\boldsymbol{\omega}}(\tau)) = \frac{\hat{r}'(\tau)}{\hat{r}(\tau)},$$

and then extending ℓ off the range of $\hat{\boldsymbol{\omega}}(\tau)$ in a smooth manner. Define the p-homogeneous vector field $\mathbf{f}$ by (2.9). Then $\hat{\boldsymbol{\omega}}(\tau)$ satisfies the system (2.3), and $\hat{r}(\tau)$ satisfies (2.4). We may take $\hat{r}(\tau)$ to be any function of τ with $\lim_{\tau\to\infty} \hat{r}'(\tau)/\hat{r}(\tau) = 0$. For instance, if we choose $\hat{r} = e^{2(\tau^2+1)^{1/4}}$, then (2.5) shows that t remains bounded, while $\hat{r}$ goes to infinity. Thus the solution blows up in finite time. If, on the other hand, we choose $\hat{r} = e^{-2(\tau^2+1)^{1/4}}$, $\hat{r}$ goes to 0, and t goes to infinity. That is $\mathbf{u}(\mathbf{t})$ goes to the origin as $t \to \infty$. If $\hat{r} \equiv 1$, we may take $\ell(\boldsymbol{\omega}) \equiv 0$, and the solution remains bounded but does not approach zero. Almost any other behavior of the solution can be obtained by a judicious choice of $\hat{r}(\tau)$. Thus all parts of Theorem 2.1 have been established. □

Example 2.1 *Consider the* 3*-dimensional* 5*-homogeneous system*

$$\begin{aligned}
\frac{du_1}{dt} &= (u_1^2+u_2^2+u_3^2)\left\{-(2u_1^2+u_2^2+3u_3^2)u_2 + [\alpha(u_1^2+u_2^2+u_3^2)+\beta u_3^2]u_1\right\} \\
\frac{du_2}{dt} &= (u_1^2+u_2^2+u_3^2)\left\{(2u_1^2+u_2^2+3u_3^2)u_1 + [\alpha(u_1^2+u_2^2+u_3^2)+\beta u_3^2]u_2\right\} \\
\frac{du_3}{dt} &= \gamma(u_1^2+u_2^2)u_3^3 + (u_1^2+u_2^2+u_3^2)[\alpha(u_1^2+u_2^2+u_3^2)+\beta u_3^2]u_3
\end{aligned} \tag{2.10}$$

where α, β, and γ are constant parameters. Then the equation (2.4) *is*

$$[\ln r]_\tau = \alpha + \beta\omega_3^2 + \gamma\omega_3^4(1-\omega_3^2), \tag{2.11}$$

while the system (2.3) *is*

$$\begin{aligned}
\frac{d\omega_1}{d\tau} &= -(2\omega_1^2+\omega_2^2+3\omega_3^2)\omega_2 - \gamma\omega_3^4(1-\omega_3^2)\omega_1 \\
\frac{d\omega_2}{d\tau} &= (2\omega_1^2+\omega_2^2+3\omega_3^2)\omega_1 - \gamma\omega_3^4(1-\omega_3^2)\omega_2 \\
\frac{d\omega_3}{d\tau} &= \gamma\omega_3^3(1-\omega_3^2)^2.
\end{aligned} \tag{2.12}$$

It is easily checked that the equilibrium points of the system (2.12) *are $\omega_1 = \omega_2 = 0$, $\omega_3 = \pm 1$. We shall assume that*

$$\gamma > 0,$$

so that all solutions for which $\boldsymbol{\omega}_3 \neq 0$ converge to one of these points.

At these points $\ell = \alpha+\beta$, Thus nonzero solutions of (2.10) *for which the initial value of u_3 is not zero blow up in a finite time if $\alpha+\beta > 0$, and go to $\mathbf{0}$ if $\alpha+\beta < 0$,*

To treat the case where $\ell = 0$, we set $\beta = -\alpha$. To find the behavior of r on such a solution, we see from (2.11) *with $\beta = -\alpha$ and from the equation for the derivative of ω_3 that*

$$\frac{d\ln r}{d\omega_3} = \frac{\alpha+\gamma\omega_3^4}{\gamma\omega_3^3(1-\omega_3^2)}.$$

If $\alpha+\gamma \neq 0$, the right-hand side behaves like $(\alpha+\gamma)\omega_3/[\gamma(1-\omega_3^2)]$ near $\omega_3 = \pm 1$, so that

$$r = O\left((1-\omega_3^2)^{-(\alpha+\gamma)/(2\gamma)}\right).$$

This shows that such solutions approach zero when $\alpha < -\gamma$, and are unbounded when $\alpha > -\gamma$. To see the degree of unboundedness, we use (2.5) *to see that*

$$\begin{aligned}\frac{dt}{d\omega_3} &= \frac{r^{-4}}{\gamma\omega_3^3(1-\omega_3^2)^2} \\ &= O\left((1-\omega_3^2)^{-2+2(\alpha+\gamma)/\gamma}\right).\end{aligned}$$

Thus, the solutions blow up in a finite time if $\alpha = -\beta > -\gamma/2$, grow up to infinity as $t \to \infty$ if $-\gamma < \alpha = -\beta \leq -\gamma/2$ and converge to the origin if $\alpha = -\beta < -\gamma$.

Another interesting class of solutions of (2.3) is the periodic solutions. We have the following result.

Theorem 2.2 *Let $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}(\tau)$ be a nonconstant solution of the system* (2.3) *on S^{d-1} which is periodic of some period Π. Define*

$$\mathit{\Gamma} := \int_0^{\Pi} \ell(\hat{\boldsymbol{\omega}}(\tau))d\tau. \tag{2.13}$$

Let $\mathbf{u}(t)$ be any solution of (1.1) *whose radial projection is either equal to or converges to the closed curve $\{\boldsymbol{\omega} : \boldsymbol{\omega} = \hat{\boldsymbol{\omega}}(\tau)\}$. Then*

a. *If $\mathit{\Gamma} > 0$, the solution $\mathbf{u}$ blows up in a finite time;*

b. *If $\mathit{\Gamma} < 0$, $\mathbf{u}$ approaches zero as $t \to \infty$;*

c. *If $\mathit{\Gamma} = 0$, then for any prescribed positive number T there is a solution $\mathbf{u}(T;t)$ of* (1.1) *which is periodic of period T in t, but a solution whose radial projection approaches the closed curve $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}$ can have any asymptotic behavior, which depends on the behavior of $\ell(\boldsymbol{\omega})$ near the closed orbit.*

Proof. We see from the equation (2.4) that if the radial projection of the solution $\mathbf{u}$ to S^{d-1} coincides with the curve $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}(t)$, then

$$\ln r(\tau + \Pi) = \ln r(\tau) + \mathit{\Gamma}, \tag{2.14}$$

so that the function $\ln r - \mathit{\Gamma}\tau/\Pi$ is Π-periodic. Therefore the same is true of the function $z(\tau) := e^{-\mathit{\Gamma}\tau/\Pi} r(\tau)$. Then (2.5) shows that

$$\frac{dt}{d\tau} = e^{-(p-1)\mathit{\Gamma}\tau/\Pi} z(\tau)^{1-p}, \tag{2.15}$$

where z is positive and Π-periodic, and hence bounded above and below by positive numbers. If $\mathit{\Gamma} > 0$, the right-hand side is thus bounded by a constant multiple of a negative exponential in τ, which implies that t remains bounded as $\tau \to \infty$. On the other hand, the recursion (2.14) shows that r goes to infinity as $\tau \to \infty$, and this proves the finite-time blowup.

If $\mathit{\Gamma} < 0$, the recursion (2.14) shows that r goes to zero as $\tau \to \infty$, while (2.15) implies that $t \to \infty$ as $\tau \to \infty$. Thus $\mathbf{u}$ approaches the origin as $t \to \infty$.

Finally, if $\mathit{\Gamma} = 0$, the recursion (2.14) shows that r is periodic of period Π in τ, and the equation (2.5) shows that

$$t(\tau + \Pi) = t(\tau) + \int_0^{\Pi} r^{1-p} d\tau.$$

It follows that $\mathbf{u}$ is periodic of period $\int_0^{\Pi} r^{1-p} d\tau > 0$ in t. We now see from Lemma 2.1 that $\mathbf{v}(t) := \rho\mathbf{u}(\rho^{p-1}t)$ is also a periodic solution, but its period is $\rho^{1-p}\int_0^{\Pi} r^{1-p} d\tau$. We can always find a value of ρ which makes this period equal to a prescribed positive value T. This completes the proof for the case in which the radial projection of $\mathbf{u}(t)$ coincides with the orbit $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}(t)$.

We now consider a solution whose radial projection converges to a periodic solution. Let Ω be the orbit of the periodic solution $\hat{\boldsymbol{\omega}}(\tau)$. Construct a neighborhood N of this closed curve on S^{d-1} with the property that, for each $\tau \in [0, \Pi)$, $\mathbf{h}(\hat{\boldsymbol{\omega}}(\tau)) \cdot \mathbf{h}(\boldsymbol{\omega}) > 0$ for all $\boldsymbol{\omega}$ in that component $P(\tau)$ of the intersection of the plane $[\boldsymbol{\omega} - \hat{\boldsymbol{\omega}}(\tau)] \cdot \mathbf{h}(\hat{\boldsymbol{\omega}}(\tau)) = 0$ with N which contains $\hat{\boldsymbol{\omega}}(\tau)$. By hypothesis, there is a τ_1 such that $\boldsymbol{\omega}(\tau) \in N$ for $\tau \geq \tau_1$. We see from the above condition that once $\boldsymbol{\omega}(\tau)$ enters and remains in N, it must follow $\hat{\boldsymbol{\omega}}$ around. By continuity, we can choose N so small that if $\boldsymbol{\omega}(\tau)$ lies on the set $P(\tau_2)$ for some τ_2, it must return to that set before τ increases by 2Π. Moreover, if $\mathit{\Gamma} > 0$, we see from (2.4) that if N is sufficiently small, $\ln r$ must increase by at least $\mathit{\Gamma}/2$ by the time $\boldsymbol{\omega}$ returns to the plane. It immediately follows that r increases to infinity exponentially as $\tau \to \infty$. Then (2.5) shows that t remains bounded, so that $\mathbf{u}$ experiences finite-time blowup. This finishes the proof of Part (a) of the Theorem.

If $\mathit{\Gamma} < 0$, the same argument shows that if N is sufficiently small, $\ln r$ decreases by at least $|\mathit{\Gamma}|/2$ whenever τ increases by some number which is at most 2Π. This shows that r goes to 0, which proves Part (b).

To prove Part (c), we again construct the neighborhood N in which $\boldsymbol{\omega}(\tau)$ follows $\hat{\boldsymbol{\omega}}(\tau)$ around. As in the proof of Part (c) of Theorem 2.1, we show that we can find an $\ell(\boldsymbol{\omega})$ such that any prescribed function $\hat{r}(\tau)$ for which $\hat{r}'/\hat{r}$ converges to zero is a solution of (2.4). The corresponding solution can have almost any behavior from finite-time blowup to convergence to the origin. ∎

Example 2.2 *We return to the 3-dimensional system* (2.10) *of Example 2.1. We see from the third equation of* (2.12) *that* ω_3 *remains fixed not only if* $\omega_3(0) = \pm 1$*, but also if* ω_3 *is initially* 0*. We then see from the first two equations that the corresponding solution of* (2.12) *has the form* $(\cos\phi(\tau), \sin\phi(\tau), 0)$ *where* $\phi(\tau)$ *solves the equation*

$$\frac{d\phi}{d\tau} = 2\cos^2\phi + \sin^2\phi.$$

It is easily seen that ϕ *is increased by* 2π *when* τ *increases by* $\Pi := \int_0^{2\pi}[2\cos^2\psi + \sin^2\psi]^{-1}d\psi$*, so that* $\boldsymbol{\omega}(\tau)$ *is periodic of period* Π.

The last equation of (2.12) *shows that for any solution whose initial value does not make the right-hand side zero,* $\omega_3(\tau)$ *converges to zero if and only if*

$$\gamma < 0.$$

We shall assume this from now on.

Since $\ell(\boldsymbol{\omega})$ *is the right-hand side of* (2.11)*, we see that on the periodic solution* $\ell = \alpha$*, so that* Γ *is just* α *times the* τ*-period. Thus Theorem 2.1 shows that if* $\alpha < 0$*, all solutions with* $u_1^2 + u_2^2 \neq 0$ *converge to the origin, which if* $\alpha > 0$*, these solutions blow up in a finite time.*

When $\alpha = 0$*,* $\Gamma = 0$*, so that* (2.10) *has periodic solutions. To study solutions whose radial projections converge to a periodic solution, we observe that when* $\alpha = 0$*,*

$$\frac{d\ln r}{d\omega_3} = \frac{\beta + \gamma\omega_3^2(1-\omega_3^2)}{\gamma\omega_3(1-\omega_3^2)^2},$$

which behaves like $\beta/(\gamma\omega_3)$ *near* $\omega_3 = 0$*. Thus when* $\beta \neq 0$*,*

$$r = O(|\omega_3|^{\beta/\gamma})$$

when ω_3 *approaches zero. Since* $\gamma < 0$*, this shows that when* $\beta < 0$*, the solutions converge to the origin, while when* $\beta > 0$*,* $|\mathbf{u}|$ *goes to infinity. Since by* (2.5)

$$\frac{dt}{d\omega_3} = \frac{r^{-4}}{\gamma\omega_3^3(1-\omega_3^2)^2},$$

which behaves like $|\omega_3|^{-3-4\beta/\gamma}$*, we find that if* $0 < \beta \leq -\gamma/2$*,* $|\mathbf{u}|$ *goes to infinity as* $t \to \infty$*, while if* $\beta > -\gamma/2$*, the solutions blow up in a finite time.*

Finally, we study a limit cycle of (2.3) and the orbits which converge to it. A limit cycle is an ordered set of zeros of $\mathbf{h}$ together with orbits of (2.3) which connect these zeros in a cyclic fashion.

Theorem 2.3 *Every solution whose orbit has a radial projection to S^{d-1} which coincides with or converges to a limit cycle converges to $\mathbf{0}$ if and only if $\ell < 0$ on the set of zeros of $\mathbf{h}$ on the limit cycle.*

Proof. We see from Theorem 2.1 that all solutions which correspond to a part of the limit cycle converge to $\mathbf{0}$ if and only if $\ell < 0$ at all equilibria on the limit cycle. Thus, this condition is necessary. On the other hand, if the condition holds, then $\int \ell d\tau$ over each of the connecting orbits of the limit cycle is $-\infty$. Then just as in the proof of Theorem 2.2 we see that $\int \ell d\tau$ over one rotation along an orbit which converges to the limit cycle will eventually be uniformly negative, and hence a corresponding solution also approaches the origin. ∎

The following criterion for the origin to be a global attractor for the system (1.1) is a direct corollary of the preceding theorems.

Theorem 2.4 *Suppose that the omega limit set of every solution of the system* (2.3) *is either an equilibrium point, a periodic solution, or a limit cycle. Then a necessary and sufficient condition for the origin to be a global attractor for the system* (1.1) *is that*

1. *the equation* (2.6) *never holds with $\ell(\boldsymbol{\omega}^*) \geq 0$, and*
2. *the constant Γ defined by* (2.13) *is strictly negative for every periodic solution of* (2.3).

Proof. Parts (a) and (c) of Theorems 2.1 and 2.2, and Theorem 2.3 show the necessity of these conditions. Parts (b) of Theorems 2.1 and 2.2, and Theorem 2.3 show their sufficiency. ∎

Remark. The hypothesis that omega limit sets contain only equilibrium points, periodic solutions, and limit cycles is automatically satisfied when $d = 2$ or 3.

3 2-dimensional p-homogeneous systems.

If the number d of independent variables is greater than 2, the flow (2.3) on the unit sphere S^{d-1} may be quite complicated. When $d = 3$, there may be equilibrium points, periodic solutions, and limit cycles. If $d > 3$, there may even be strange attractors. It is even difficult to resolve cases where $\ell(\boldsymbol{\omega}^*) = 0$ at a zero of $\mathbf{h}$ or where $\Gamma = 0$ on a periodic solution, except for some special problems such as the one found in the Examples of Section 2. The situation is simpler when $d = 2$, because the unit circle S^1

is a simpler set than S^{d-1} with $d > 1$. We shall investigate this simplification in this Section.

We consider the system

$$\begin{aligned} \frac{du_1}{dt} &= f_1(u_1, u_2) \\ \frac{du_2}{dt} &= f_2(u_1, u_2), \end{aligned} \tag{3.1}$$

where the functions f_1 and f_2 are p-homogeneous in (u_1, u_2). The unit circle S^1 is easily parametrized in terms of a single variable θ by setting

$$\boldsymbol{\omega} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Since the vector field $\mathbf{h}(\boldsymbol{\omega})$ is orthogonal to $\boldsymbol{\omega}$, it can be written in the form

$$\mathbf{h}(\cos\theta, \sin\theta) = h(\theta) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

The scalar function $h(\theta)$ is easily seen to be given by the formula

$$h(\theta) = -f_1(\cos\theta, \sin\theta)\sin\theta + f_2(\cos\theta, \sin\theta)\cos\theta.$$

Then the system (2.3) becomes the single equation

$$\theta_\tau = h(\theta). \tag{3.2}$$

If we abuse the notation by writing $\ell(\theta)$ instead of $\ell(\cos\theta, \sin\theta)$, then

$$\ell(\theta) = f_1(\cos\theta, \sin\theta)\cos\theta + f_2(\cos\theta, \sin\theta)\sin\theta,$$

and the equation (2.4) becomes

$$[\ln r]_\tau = \ell(\theta). \tag{3.3}$$

The equation (2.5) remains unchanged. We observe that the functions h and ℓ are both periodic of period 2π. We are now ready to state a more complete version of Theorems 2.1, and 2.2.

Theorem 3.1 *1. If $h(\theta)$ has at least one zero, then*

- *a. all solutions of the system* (3.1) *converge to* $(0,0)$ *if and only if $\ell(\theta_*) < 0$ whenever $h(\theta_*) = 0$.*
- *b. There is an unbounded solution of* (3.1) *if and only if there is a zero θ_* of h for which either $\ell(\theta_*) > 0$, or $\ell(\theta_*) = 0$ and there is a number $\theta_0 \neq \theta_*$ such that $(\theta_* - \theta_0)h(\theta) > 0$ at θ_0 and at all points between θ_0 and θ_*, and, if $\ell(\theta_*) = 0$, the function*

$$I(\theta_0, \theta) := \int_{\min\{\theta_0,\theta\}}^{\max\{\theta_0,\theta\}} \frac{\ell(\phi)}{|h(\phi)|} d\phi \tag{3.4}$$

is not bounded from above.

c. *There is a solution which stops existing after a finite time if and only if for some zero θ_* of h the condition of Statement (b) is satisfied, and the integral*

$$\int_{\min\{\theta_0,\theta_*\}}^{\max\{\theta_0,\theta_*\}} \frac{e^{-(p-1)I(\theta_0,\psi)}}{|h(\psi)|} d\psi$$

is finite. This can only happen if $I(\theta_0, \theta)$ is unbounded at θ_*. If in addition, $I(\theta_0, \theta)$ converges to infinity as θ approaches θ_*, then the solution $r\boldsymbol{\omega}$ blows up in a finite time.

2. *If $h(\theta)$ is never zero, then every solution of* (3.2) *is periodic. As in Section 2, let*

$$\mathit{\Gamma} = \int_0^{2\pi} \frac{\ell(\theta)}{|h(\theta)|} d\theta$$

be the increase in $\ln r$ *during one period, Then if $\mathit{\Gamma} > 0$, all solutions of* (3.1) *blow up in a finite time, if $\mathit{\Gamma} < 0$, all solutions of* (3.1) *approach zero, and if $\mathit{\Gamma} = 0$, all solutions of* (3.1) *are periodic.*

Proof. Let (r_0, θ_0) be the values of a solution at $t = 0$. We first consider the case where $h(\theta_0) \neq 0$. We see from (3.2) that if $h(\theta_0) > 0$, then θ increases to the nearest zero θ_* of h. If $h(\theta_0) < 0$, then $\theta(\tau)$ decreases to the nearest zero of h. In either case we see that $(\theta_* - \theta_0)h(\theta) > 0$ at θ_0 and for all θ between θ_0 and θ_*. Since $d\theta/d\tau \neq 0$ on this set, we can write

$$\frac{d(\ln r)}{d\theta} = \frac{(\ln r)_\tau}{\theta_\tau} = \frac{\ell(\theta)}{h(\theta)}$$

We integrate this equation to obtain the fact that

$$\ln r = \ln r_0 + I(\theta_0, \theta),$$

where the function I is defined by (3.4). Thus

$$r = r_0 e^{I(\theta_0,\theta)}. \tag{3.5}$$

We can thus express r in terms of the parameter θ. In order to have a parametric solution of (3.1), we also need to express t as a function of θ. We see from the equation $\theta_t = r^{p-1}h(\theta)$ that

$$t = \int_{\min\{\theta_0,\theta\}}^{\max\{\theta_0,\theta\}} \frac{r_0^{-(p-1)} e^{-(p-1)I(\theta_0,\psi)}}{|h(\psi)|} d\psi. \tag{3.6}$$

All the statements of Part (1) of the Theorem can be read off from (3.5) and (3.6) when $h(\theta_0) \neq 0$, and from Theorem 2.1 when $h(\theta_0) = 0$.

In order to prove Part (2) of the Theorem, we observe that when h has no zeros, the parametric solution given by (3.5) and (3.6) is defined for all initial values θ_0 and for all θ. Moreover, because $\ell(\phi)/|h(\phi)|$ is periodic of period 2π, we see that $I(\theta_0, \theta)$ increases by Γ whenever $h(\theta_0)\theta$ is increased by $2\pi h(\theta_0)$. Thus, if $\Gamma > 0$, I increases linearly in $|\theta|$, so that r is unbounded, and (3.6) shows that r blows up in finite time. If $\Gamma < 0$, the same reasoning shows that r goes to zero. Finally, if $\Gamma = 0$, then $I(\theta_0, \theta)$ is periodic of period 2π in θ. Hence, r is a periodic function of θ, and the period of $r(t)$ in t is obtained by setting $\theta = \theta_0 + 2\pi$ in (3.6). Thus the proof of Theorem 3.1 is complete. □

The following example shows that $|\mathbf{u}(t)|$ need not converge to infinity on an unbounded solution.

Example 3.1 *Consider the 7-homogeneous system*

$$\frac{du_1}{dt} = \left[(u_1^2 + u_2^2)\sin(\sqrt{u_1^2 + u_2^2}/u_2) - u_2\sqrt{u_1^2 + u_2^2}\cos(\sqrt{u_1^2 + u_2^2}/u_2)\right]u_1^2 u_2^3 - u_2^7$$
$$\frac{du_2}{dt} = \left[(u_1^2 + u_2^2)\sin(\sqrt{u_1^2 + u_2^2}/u_2) - u_2\sqrt{u_1^2 + u_2^2}\cos(\sqrt{u_1^2 + u_2^2}/u_2)\right]u_1 u_2^4 + u_1 u_2^6.$$

Then

$$\ell(\theta) = \left[\sin\left(\frac{1}{\sin\theta}\right) - \sin\theta\cos\left(\frac{1}{\sin\theta}\right)\right]\cos\theta\sin^3\theta,$$

and

$$h(\theta) = \sin^6\theta.$$

Since $h \geq 0$ and h has zeros at 0 and π, we see that if $0 < \theta_0 < \pi$, then $\theta(t)$ increases to π. If $-\pi < \theta_0 < 0$, then θ increases to 0. It is easily verified that

$$I(\theta_0, \theta) = \frac{\cos(1/\sin\theta)}{\sin\theta} - \frac{\cos(1/\sin\theta_0)}{\sin\theta_0}.$$

As θ approaches either π or 0, the formula $r = r_0 e^{I(\theta_0,\theta)}$ shows that $r(t(\theta))$ has the limit superior ∞ and the limit inferior 0.

4 Linear perturbations of p-homogeneous systems.

In this section we shall consider systems of the form

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}) + \alpha A\mathbf{u} \tag{4.1}$$

where $\mathbf{u}$ and $\mathbf{f}$ are d-vector-valued functions, $\mathbf{f}(\mathbf{u})$ is positive homogeneous of degree $p > 1$ in $\mathbf{u}$, and A is a constant matrix. We are interested in the large-time asymptotics of solutions of this equation. We begin with the special case where A is the identity matrix, so that the linear perturbation is purely radial. That is, we are looking at the family of equations

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}) + \alpha\mathbf{u} \tag{4.2}$$

with α a constant. In this case we have a rather complete result.

Theorem 4.1 *1. If for one value of α all solutions of the system* (4.2) *exist for all time, then the same is true of this equation for any other value of α. Moreover, in this case, all solutions of* (4.2) *with $\alpha < 0$ approach the origin as $t \to \infty$.*

2. If the equation (4.2) *with one value of α has a solution which blows up in a finite time, then the equation* (4.2) *with any α has such a solution.*

3. If the system (4.2) *with $\alpha = 0$ has a solution which does not converge to $\mathbf{0}$ as $t \to \infty$, then for every positive α the equation* (4.2) *has an unbounded solution.*

Proof. It is easily verified that if $\mathbf{u}^{(0)}(t)$ satisfies the equation (4.2) with $\alpha = 0$, then for any α the function

$$\mathbf{u}^{(\alpha)}(t) := e^{\alpha t}\mathbf{u}^{(0)}\left(\frac{e^{\alpha(p-1)t} - 1}{\alpha(p-1)}\right) \tag{4.3}$$

is the solution of the equation (4.2) with the same initial value as $\mathbf{u}$.

The elementary theory of differential equations shows that a solution of (4.2) ceases to exist at a finite value t_B of t if and only if $\limsup_{t \nearrow t_B} |\mathbf{u}(t)| = \infty$. Thus if $u^{(0)}$ ceases to exist at time t_B, we see from the fact that the argument of $\mathbf{u}^{(\alpha)}$ in (4.3) goes to infinity with t when $\alpha > 0$ and to $1/[(-\alpha)(p-1)]$ when $\alpha < 0$ that $\mathbf{u}^{(\alpha)}$ also ceases to exist in a finite time as long as

$$\alpha > -\frac{1}{(p-1)t_B}.$$

We see from Lemma 2.1 that the function $\mathbf{v}^{(0)} := \rho\mathbf{u}^{(0)}(\rho^{p-1}t)$ is again a solution of the equation with $\alpha = 0$. If $\mathbf{u}^{(0)}$ ceases to exist at time t_B, then this function ceases to exist at time $\rho^{1-p}t_B$. Then the solution of (4.2) defined by (4.3) with $\mathbf{u}^{(0)}$ replaced by $\mathbf{v}^{(0)}$ ceases to exist in finite time, provided $\alpha > -\rho^{p-1}/(p-1)t_B$. For any given α, this inequality is true when ρ is chosen to be sufficiently large. Thus if (4.2) with $\alpha = 0$ has a solution which ceases to exist in finite time, the same is true of (4.2) with any α.

It follows that if all solutions of (4.2) with some value α_0 of α are defined for all $t > 0$, the same is true of the equation with $\alpha = 0$. We see from this and (4.3) that for any α all solutions of (4.2) are defined for all t. Moreover, we observe that if $\alpha < 0$, the argument of $\mathbf{u}^{(0)}$ in (4.3) remains bounded, while the factor in front approaches zero. Thus every solution of (4.2) with $\alpha < 0$ approaches zero. This is Statement (1).

Since a solution blows up at a finite time t_B if and only it ceases to exist at t_B and r goes infinity as t increases to t_B, the above argument can be extended by replacing "ceases to exist" by "blows up". This gives Statement (2).

Finally, we observe that if $\alpha > 0$ and $\limsup_{t\to\infty} |\mathbf{u}^{(0)}(t)| > 0$, then by (4.3)

$$\limsup_{t\to\infty} |\mathbf{u}^{(\alpha)}(t)| = \limsup_{t\to\infty}\left[e^{\alpha t}\left|\mathbf{u}^{(0)}\left(\frac{e^{\alpha(p-1)t} - 1}{\alpha(p-1)}\right)\right|\right] = \infty,$$

which is Statement (3). This completes the proof of Theorem 4.1. $\square$

Remark. This Theorem says that the property that all solutions are globally defined is not changed by the addition of a multiple of $\mathbf{u}$ to the right-hand side of a p-homogeneous system. In contrast, the system (2) of [2] with $\alpha = \beta = 0$ is an example of a nonhomogeneous system all of whose solutions converge to the origin, for which adding any negative multiple of $\mathbf{u}$ to the right-hand side produces solutions which blow up in a finite time.

One might expect to be able to extend the result of Theorem 4.1 to matrices A with the property that $\boldsymbol{\omega} \cdot A\boldsymbol{\omega} > 0$ for all $\boldsymbol{\omega} \in S^{d-1}$. This property says that the radial component of $A\boldsymbol{\omega}$ is always positive. In fact, one can obtain the following Proposition.

Proposition 4.1 *Suppose that the function* $\mathbf{f}$ *has the property*

$$\boldsymbol{\omega} \cdot \mathbf{f}(\boldsymbol{\omega}) \leq 0 \text{ for all } \boldsymbol{\omega} \in S^{d-1}. \tag{4.4}$$

Also suppose that

$$\boldsymbol{\omega} \cdot A\boldsymbol{\omega} > 0 \text{ for all } \boldsymbol{\omega} \in S^{d-1}. \tag{4.5}$$

Then the origin is a global attractor for the system (4.1) *when* $\alpha < 0$*, but not when* $\alpha > 0$*.*

Proof. We see from (4.5) and continuity that there is a positive constant σ such that

$$\mathbf{u} \cdot A(\mathbf{u}) \geq \sigma |\mathbf{u}|^2$$

for all $\mathbf{u}$. The equation (4.1), this inequality, and (4.4) imply that when $\alpha < 0$,

$$\frac{1}{2} \frac{d|\mathbf{u}|^2}{dt} \leq \alpha\sigma |\mathbf{u}|^2,$$

so that $|\mathbf{u}|^2$ approaches zero exponentially.

When $\alpha > 0$, it is easily seen that the real parts of all the eigenvalues of αA are positive, so that the origin is a repeller rather than attractor. This proves the Proposition. ∎

The trouble with this proposition is that, as Theorem 2.4 shows, the system (1.1) can have the origin as a global attractor for many $\mathbf{f}$ for which the condition (4.4) is not satisfied. We shall show that if the condition (4.4) is violated, that is, if there is a unit vector $\hat{\boldsymbol{\omega}}$ so that

$$\hat{\boldsymbol{\omega}} \cdot \mathbf{f}(\hat{\boldsymbol{\omega}}) > 0, \tag{4.6}$$

then there is a matrix A which satisfies (4.5) such that the origin is not globally stable for (4.1) with $\alpha < 0$. In order to state this result, we define the skew-symmetric matrix S by defining its entries

$$S_{ij} := f_i(\hat{\boldsymbol{\omega}})\hat{\omega}_j - \hat{\omega}_i f_j(\hat{\boldsymbol{\omega}}). \tag{4.7}$$

Proposition 4.2 *Suppose that there is at least one unit vector $\hat{\boldsymbol{\omega}}$ for which the inequality* (4.6) *is satisfied, so that the inequality* (4.4) *is violated. Define the matrix S by* (4.7)*. Then the matrix*

$$A := \hat{\boldsymbol{\omega}} \cdot \mathbf{f}(\hat{\boldsymbol{\omega}}) I + S \tag{4.8}$$

has the property (4.5)*, but the origin is not a global attractor for the system* (4.1) *for any $\alpha \neq 0$.*

Proof. It is easily seen that $\boldsymbol{\omega} \cdot A\boldsymbol{\omega} = \hat{\boldsymbol{\omega}} \cdot \mathbf{f}(\hat{\boldsymbol{\omega}}) > 0$. Therefore A has the property (4.5). We also see from the definitions of A and S that

$$A\hat{\boldsymbol{\omega}} = \mathbf{f}(\hat{\boldsymbol{\omega}}).$$

The p-homogeneity of $\mathbf{f}$ then shows that if $\alpha < 0$, then the vector

$$\hat{\mathbf{u}} := (-\alpha)^{1/(p-1)} \hat{\boldsymbol{\omega}}$$

satisfies the equation

$$\mathbf{f}(\hat{\mathbf{u}}) + \alpha A \hat{\mathbf{u}} = \mathbf{0}.$$

That is, $\hat{\mathbf{u}}$ is an equilibrium point of (4.1). Since the solution of (4.1) which starts at $\hat{\mathbf{u}}$ does not go to the origin, the origin is not a global attractor.

When $\alpha > 0$, the skew symmetry of S shows that the real parts of all the eigenvalues of αA are equal to $\hat{\boldsymbol{\omega}} \cdot \mathbf{f}(\hat{\boldsymbol{\omega}}) > 0$. We conclude that the origin is a repeller and therefore not a global attractor. Thus the statement of the Proposition is established. ∎

Example 4.1 *Consider the two-dimensional* 3*-homogeneous system*

$$\begin{aligned} \frac{du_1}{dt} &= u_1^3 - u_1^2 u_2 - u_1 u_2^2 \\ \frac{du_2}{dt} &= u_1^3 + u_1^2 u_2 - u_2^3. \end{aligned} \tag{4.9}$$

Then the right-hand side of (3.2) *is*

$$h(\theta) = f_2(\cos\theta, \sin\theta)\cos\theta - f_1(\cos\theta, \sin\theta)\sin\theta = \cos^2\theta, \tag{4.10}$$

and the right-hand side of (3.3) *is*

$$\ell(\theta) = f_1(\cos\theta, \sin\theta)\cos\theta + f_2(\cos\theta, \sin\theta)\sin\theta = \cos^2\theta - \sin^2\theta. \tag{4.11}$$

The equilibria of (3.2) *are $\theta_* = \pm\pi/2$. At these points ℓ is negative. There are no periodic solutions of* (3.2)*, and so by Theorem 2.4 the origin is a global attractor for* (4.9)*. On the other hand, the condition* (4.6) *is clearly satisfied for $\hat{\theta} = 0$. The matrix* (4.8) *is then*

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Thus Theorem 4.2 states that the origin is not a global attractor for the system

$$\begin{aligned}\frac{du_1}{dt} &= u_1^3 - u_1^2 u_2 - u_1 u_2^2 + \alpha(u_1 - u_2)\\ \frac{du_2}{dt} &= u_1^3 + u_1^2 u_2 - u_2^3 + \alpha(u_1 + u_2).\end{aligned} \tag{4.12}$$

for any α other than 0.

We see from Theorem 4.1 that the asymptotic behaviors of the solutions of the system (4.2) with $\alpha = \alpha_1$ and $\alpha = \alpha_2$ are the same as long as α_1 and α_2 have the same signs. The following Lemma shows that this property is also valid for the more general system (4.1). It is proved by a simple application of the chain rule and the p-homogeneity of $\mathbf{f}$.

Lemma 4.1 *If $\mathbf{u}^{(\alpha_1)}$ is a solution of the equation* (4.1) *with $\alpha = \alpha_1$ and if $\alpha_1\alpha_2 > 0$, then*

$$\mathbf{u}^{(\alpha_2)}(t) := \left(\frac{\alpha_2}{\alpha_1}\right)^{1/(p-1)} \mathbf{u}^{(\alpha_1)}\left(\frac{\alpha_2 t}{\alpha_1}\right)$$

is a solution of (4.1) *with $\alpha = \alpha_2$.*

This Lemma shows that the asymptotic properties of solutions are the same for any two values of α which have the same sign. Thus, it is sufficient to consider the three values $\alpha = -1,\ 0,$ and 1. In particular, the size of α does not matter, so that for some purposes α cannot be considered a perturbation parameter. One must, instead, look at A as perturbation of a scalar matrix σI, for which we have the properties of Theorem 4.1.

5 A linear perturbation result in the two-dimensional case.

It seems to be difficult to obtain a perturbation result for a system of more than two dependent variables. This is partly due to the difficulty of classifying flows on S^{d-1} when $d > 2$. As we have already seen in section 3, the unit circle S^1 is easily parametrized by an angle variable θ by setting $\boldsymbol{\omega} = (\cos\theta, \sin\theta)$. The system

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}) + \alpha A \mathbf{u} \tag{5.1}$$

can be written in the form

$$\begin{aligned}\frac{d\ln r}{dt} &= r^{p-1}\ell(\theta) + \alpha P(\theta)\\ \frac{d\theta}{dt} &= r^{p-1}h(\theta) + \alpha Q(\theta),\end{aligned} \tag{5.2}$$

where, as before,

$$\ell(\theta) = \mathbf{f}_1(\cos\theta, \sin\theta)\cos\theta + \mathbf{f}_2(\cos\theta, \sin\theta)\sin\theta,$$
$$h(\theta) = -\mathbf{f}_1(\cos\theta, \sin\theta)\sin\theta + \mathbf{f}_2(\cos\theta, \sin\theta)\cos\theta,$$

while

$$P(\theta) = (\cos\theta, \sin\theta) \cdot A \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$
$$Q(\theta) = (-\sin\theta, \cos\theta) \cdot A \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Thus P and Q are homogeneous polynomials of degree two in $\cos\theta$ and $\sin\theta$.

Theorem 5.1 *Suppose that the origin is a global attractor for the two-dimensional system $\mathbf{u}_t = \mathbf{f}(\mathbf{u})$, and that the matrix A satisfies the condition*

$$\boldsymbol{\omega} \cdot A\boldsymbol{\omega} \geq \sigma > 0 \quad \text{for } \boldsymbol{\omega} \in S^1. \tag{5.3}$$

Then there exists a positive constant ε which depends only on properties of the functions $\ell(\theta)$ and $h(\theta)$ such that the inequality

$$|Q(\theta)| \leq \sigma\varepsilon \quad \text{for all } \theta,$$

implies that the origin is a global attractor for the system (5.1) when $\alpha < 0$, but not when $\alpha > 0$.

Proof. Because the inequality $\boldsymbol{\omega} \cdot A\boldsymbol{\omega} \geq \sigma$ implies that the real parts of all eigenvalues of A are at least σ, the origin is a repeller when $\alpha > 0$.

We see from Lemma 4.1 that the behavior of the solutions when $\alpha < 0$ is the same as that for $\alpha = -1$. Therefore we shall study only this case. Proposition 4.1 shows that if $\ell \leq 0$ for all θ, the origin is a global attractor, so that the conclusion of the Theorem is true. Hence we assume without loss of generality that

$$\ell_{\max} := \max[\ell(\theta)] > 0.$$

We write (5.1) in the the polar form (5.2). The condition (5.3) shows that $P(\theta) \geq \sigma > 0$. We see from the first equation of (5.2) that if R is a positive number such that

$$R^{p-1} < \frac{\sigma}{\ell_{\max}} \tag{5.4}$$

then $(\ln r)_t$ is uniformly negative for $r \leq R$. Thus the set $r \leq R$ belongs to the basin of attraction of the origin. Consequently, a solution $\mathbf{u}$ which does not approach the origin must satisfy $r > R$ for all t. We consider two cases:

Case 1. Suppose that $h(\theta)$ is never zero. Then $|h| \geq |h|_{\min} > 0$, and when $\alpha = 0$, every solution of (3.2) is periodic. We see from Theorem 2.2 that because the origin is a global attractor for this equation,

$$\Gamma = \int_0^{2\pi} \frac{\ell(\theta)}{|h(\theta)|} d\theta < 0.$$

Assume for the time being that h is positive, so that $h \geq |h|_{\min}$. We see from the θ-equation in (5.2) that if μ is any constant with $0 < \mu < 1$ and we require that

$$|Q(\theta)| \leq \mu R^{p-1} |h|_{\min}, \tag{5.5}$$

then as long as r remains above R, θ increases at a uniformly positive speed.

We see from (5.2) that

$$\frac{d \ln r}{d\theta} = \frac{r^{p-1}\ell(\theta) - P(\theta)}{r^{p-1}h(\theta) - Q(\theta)}.$$

Thus

$$\ln r(2\pi[k+1]) - \ln r(2\pi k) = \int_{2\pi k}^{2\pi[k+1]} \frac{(\ell/h) - r^{1-p}(P/h)}{1 - r^{1-p}(Q/h)} d\theta.$$

By (5.5) the denominator is bounded below by $1 - \mu$ when $r \geq R$. Since $P \geq 0$ and $h > 0$, we then obtain the upper bound

$$\begin{aligned} \ln r(2\pi[k+1]) - \ln r(2\pi k) &< \int_{2\pi k}^{2\pi[k+1]} \frac{(\ell/h)}{1 - r^{1-p}(Q/h)} d\theta \\ &= \int_{2\pi k}^{2\pi[k+1]} \frac{\ell}{h} \left\{ 1 + \frac{r^{1-p}(Q/h)}{1 - r^{1-p}(Q/h)} \right\} d\theta \\ &\leq \int_{2\pi k}^{2\pi[k+1]} \left\{ \frac{\ell}{h} + \frac{\mu}{1-\mu} \frac{|\ell|}{h} \right\} d\theta. \end{aligned}$$

Thus if μ is so small that

$$\frac{\mu}{1-\mu} \int_{2\pi k}^{2\pi[k+1]} \frac{|\ell|}{h} d\theta < -\Gamma, \tag{5.6}$$

then $\ln r(2\pi[k+1]) - \ln r(2\pi k)$ is bounded above by a negative number which is independent of k. By summing the bound for the difference of $\ln r$ with respect to k, we see that for any solution for which $r(0) > R$, $\ln r$ becomes equal to $\ln R$ in a finite time. As we have seen, r then converges to 0.

In view of the definition (5.5) of μ, the inequality (5.6) is equivalent to

$$\max |Q| < R^{p-1} |h|_{\min} \frac{-\displaystyle\int_0^{2\pi} (\ell/h) d\theta}{\displaystyle\int_0^{2\pi} ([|\ell| - \ell]/h) d\theta}.$$

We satisfy the inequality (5.4) by choosing $R = [\sigma/(2\ell_{\max})]^{1/(p-1)}$. Thus we have proved the Theorem with

$$\varepsilon = |h|_{\min} \frac{-\int_0^{2\pi} (\ell/h) d\theta}{2\ell_{\max} \int_0^{2\pi} ([|\ell| - \ell]/h) d\theta}$$

for the case $h > 0$.

When $h < 0$ everywhere, the change of variable $\phi = -\theta$ gives a system of the form (5.2) with $\ell(\theta)$ and $P(\theta)$ replaced by $\ell(-\phi)$ and $P(-\phi)$, and with $h(\theta)$ and $Q(\theta)$ replaced by $-h(-\phi)$ and $-Q(-\phi)$. Since $-h = |h| > 0$, the above proof gives the same result with h replaced by $|h|$ in the formula for ε. Thus we have established the Theorem when h is never zero.

Case 2. Suppose that $h(\theta)$ has at least one zero. By Theorem 2.4 and the hypothesis that the origin is a global attractor for the equation (5.2) with $\alpha = 0$, we see that $\ell(\theta) < 0$ on the closed bounded set $\mathcal{N} := \{\theta : h(\theta) = 0\}$. Define

$$\gamma := -\frac{1}{2} \max_{\theta \in \mathcal{N}} \ell(\theta) > 0,$$

the open set

$$\mathcal{S}_\gamma := \{\theta : \ell(\theta) < -\gamma\},$$

and the open set

$$\hat{\mathcal{S}}_\gamma := \bigcup \{\text{components } C \text{ of } \mathcal{S}_\gamma \text{ which intersect } \mathcal{N}\}.$$

Then $\hat{\mathcal{S}}_\gamma$ consists of finitely many disjoint open intervals and contains $\mathcal{N}$. Therefore $|h| > 0$ on the closed bounded set $S^1 \setminus \hat{\mathcal{S}}_\gamma$. Since $\ell_{\max} > 0$, this closed bounded set is not empty. Hence $|h|$ takes on its minimum

$$\eta := \min_{\theta \notin \hat{\mathcal{S}}_\gamma} |h(\theta)| > 0$$

on this set. We assume that

$$|Q(\theta)| \leq \frac{R^{p-1}\eta}{2}. \tag{5.7}$$

Then the second equation of (5.2) shows that when $\theta(t) \in S^1 \setminus \hat{\mathcal{S}}_\gamma$, it either increases or decreases at a speed at least $R^{p-1}\eta/2$, depending on the sign of h there. Thus θ reaches a component of $\mathcal{S}_\gamma$ in a uniformly bounded time. θ may remain in this component, or, if h has the same sign on both sides of the component, it may leave this component and enter another one in a uniformly bounded time, and so forth. Because $P > 0$,

$$\frac{dr^{1-p}}{dt} = -(p-1)(\ell - r^{1-p}P) \geq (p-1)\gamma \quad \text{in } \hat{\mathcal{S}}_\gamma.$$

Thus if θ remains in one component sufficiently long, then r reaches the value R, after which we know that r decreases to 0.

If h takes on both positive and negative values outside $\hat{\mathcal{S}}_\gamma$, then at least one of the components has the property that h is positive at its left boundary point and negative at its right boundary. We then see from (5.7) that the same is true of $d\theta/dt$. This means that once θ enters this component, it cannot leave it, so that r is eventually reduced to R, and then converges to zero. It is easily seen that no matter where θ starts, it may keep entering and exiting components, but eventually either r falls to R or θ hits the component with the above property and then stays there. Thus in all cases r goes to zero if (5.7) is valid and h takes on both positive and negative values outside $\hat{\mathcal{S}}_\gamma$.

Suppose now that h has only one sign on the complement of $\hat{\mathcal{S}}_\gamma$. Then when θ leaves one component it always increases or always decreases to the next component. We consider two possibilities:

a. If $h(\theta)$ takes on both positive and negative values, and if $\hat{\theta}$ is outside $\hat{\mathcal{S}}_\gamma$, then the function $\mathrm{sgn}[h(\hat{\theta})]h(\theta)$ takes on a negative minimum value at some point $\theta_{\min}$ of $\hat{\mathcal{S}}_\gamma$. If

$$|Q(\theta)| \leq \frac{1}{2}R^{p-1}|h(\theta_{\min})|,$$

we see that $\mathrm{sgn}[h(\hat{\theta})]\,(h(\theta_{\min}) - r^{1-p}Q(\theta_{\min})) < 0$. Hence θ cannot get beyond $\theta_{\min}$ when $r \geq R$, and hence gets stuck in some component of $\hat{\mathcal{S}}_\gamma$. We again conclude that r is eventually reduced to R, and hence approaches 0.

b. The alternative is that h has at least one zero θ_*, but never changes sign. Then $h(\theta_*) = h'(\theta_*) = 0$. Choose a positive δ so small that the interval $|\theta - \theta_*| \leq \delta$ lies in $\hat{\mathcal{S}}_\gamma$. Because h is twice continuously differentiable, there is a constant c such that

$$|h(\theta)| \leq c\delta^2 \quad \text{for } |\theta - \theta_*| \leq \delta.$$

If we require Q to satisfy

$$|Q(\theta)| \leq \frac{1}{2}R^{p-1}c\delta^2,$$

then $d\ln r/d|\theta| \leq \ell/|h - R^{1-p}Q| \leq -2\gamma/(c\delta^2)$ on a θ-interval of length 2δ. We conclude that $\ln r$ is reduced by at least $4\gamma/(c\delta)$ if θ traverses the component which contains θ_*. By the same reasoning we see that $\ln r$ cannot increase by more than $2\pi\ell_{\max}/\eta$ when θ increases by 2π. Thus if

$$\delta < \frac{2\gamma\eta}{\pi c\ell_{\max}},$$

then if θ manages to increase by 2π by not getting stuck in a component of $\hat{\mathcal{S}}_\gamma$, $\ln r$ is decreased by a fixed amount. Thus on any orbit whose θ does not get stuck in a component of $\hat{\mathcal{S}}_\gamma$, $r(t)$ is eventually reduced to the value R, after which r converges to 0.

We have shown in all sub-cases of Case 2 that if Q satisfies an inequality of the form $|Q(\theta)| \leq R^{1-p}\rho$ where ρ depends only on properties of ℓ and h, then the origin is a global attractor. Choose ρ as either $\eta/2$, or $\min\{\eta/2, |h(\theta_{\min})|/2\}$ or $\min\{\eta/2, c[\gamma\eta/(\pi c\ell_{\max})]^2/2\}$, depending on the situation. Choose $R = [\sigma/(2\ell_{\max})]^{1/(p-1)}$, so that (5.4) is satisfied. In this way we find that the statement of the Theorem is satisfied when $\varepsilon = \rho/(2\ell_{\max})$. The Theorem has been established in all cases. ∎

6 Linear perturbation can produce blowup.

As we shall show, the following Theorem can be used to prove that a linear perturbation can take a system all of whose solutions are bounded to one which has solutions which blow up in a finite time.

Theorem 6.1 *Assume that $h(\theta_*) = \ell(\theta_*) = 0$. Also suppose that either*

a. *there is a $\theta_+ > \theta_*$ such that*

   i. *$\ell(\theta) > 0$ and $h'(\theta) \neq 0$ for $\theta_* < \theta < \theta_+$;*

   ii. *the functions ℓ/h' and h/h' are continuous on the closed interval $[\theta_*, \theta_+]$ and right-differentiable at θ_*;*

   iii. *if $\nu_* := [\ell/h'](\theta_*)$, then $(p-1)\nu_* + 1 > 0$;*

   iv. *$[h/h'](\theta_*) = 0$ and $[h/h']'(\theta_*) > 0$; and*

   v. *$\nu_* Q(\theta_*) > 0$*

   or

b. *there is a $\theta_- < \theta_*$ such that $\ell(\theta) > 0$ and $h'(\theta) \neq 0$ in the interval (θ_-, θ_*), (ii) is valid with $[\theta_*, \theta_+]$ replaced by $[\theta_-, \theta_*]$ and "right-differentiable" replaced by "left-differentiable", (iii) is valid, and the inequalities in (iv) and (v) are reversed.*

Then the system (5.1) *with $\alpha = -1$ has at least one solution which blows up in a finite time.*

Proof. We first suppose that the assumptions of Part (a) are valid. In order to find an orbit which corresponds to a solution which blows up, we introduce the new dependent variable

$$\mu := r^{p-1} h(\theta) \quad \text{in } (\theta_*, \theta_+),$$

and a new independent variable σ such that

$$\frac{d\sigma}{dt} = \frac{h'}{h}.$$

The system (5.2) with $\alpha = -1$ then becomes

$$\begin{aligned} \mu_\sigma &= \mu \left[\frac{(p-1)(\mu\ell - Ph)}{h'} + \mu - Q \right], \\ \theta_\sigma &= \frac{h}{h'}(\mu - Q). \end{aligned} \tag{6.1}$$

By our hypotheses, this system has the equilibrium (μ_*, θ_*), where

$$\mu_* := \frac{Q(\theta_*)}{(p-1)\nu_* + 1}. \tag{6.2}$$

An easy calculation shows that the Jacobian matrix of the right-hand sides at (μ_*, θ_*) is

$$\begin{pmatrix} Q(\theta_*) & c \\ 0 & \frac{-(p-1)\nu_* Q(\theta_*)}{(p-1)\nu_*+1}[h/h']'(\theta_*) \end{pmatrix}$$

with some constant c.

We see from the assumptions (iii), (iv), and (v) of Part (a) that the lower right entry is negative, and that the upper left entry is not zero. Then the point (μ_*, θ_*) is either a saddle point or an attractor. In either case, there is at least one trajectory which converges to this equilibrium. Since μ_* is finite and $h(\theta_*) = 0$, $r(t)$ must converge to infinity on the corresponding solution of (5.2). We see from the θ-equation of (5.2) with $\alpha = -1$ that

$$\frac{dt}{d\theta} = \frac{1}{\mu - Q(\theta)}.$$

Since the right-hand side converges to a finite negative number as θ decreases to θ_*, we conclude that θ becomes θ_* in a finite time, so that the solution blows up in a finite time. This proves the Theorem under the assumptions of Part (a). If the assumptions of Part (b) are satisfied, we replace the variable θ by $2\theta_* - \theta$ to obtain a new problem which satisfies the assumptions of Part (a). Thus the Theorem is established. □

The following application of this Theorem shows that an arbitrarily small linear perturbation of a p-homogeneous system all of whose solutions are bounded can produce solutions which blow up.

Example 6.1 *Consider the system*

$$\begin{aligned} \frac{du_1}{dt} &= (u_2 - u_1)(u_1^2 + 2u_1u_2 - u_2^2) + \alpha(1-\varepsilon)u_1 \\ \frac{du_2}{dt} &= (u_2 - u_1)(-u_1^2 + 2u_1u_2 + u_2^2) + \alpha(1+\varepsilon)u_2, \end{aligned} \tag{6.3}$$

which can be written in the polar form

$$\begin{aligned} \frac{d\ln r}{dt} &= r^2(\sin\theta - \cos\theta)(\sin\theta + \cos\theta) + \alpha(1 - \epsilon[\cos^2\theta - \sin^2\theta]) \\ \frac{d\theta}{dt} &= (\sin\theta - \cos\theta)^2 + 2\alpha\epsilon \sin\theta\cos\theta. \end{aligned}$$

When $\alpha = 0$, one obtains the equation

$$\frac{d\ln r}{d\theta} = \frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}.$$

Integration shows that if $\sin\theta_0 - \cos\theta_0 \neq 0$*, then*

$$r = \frac{r_0}{\sin\theta_0 - \cos\theta_0}(\sin\theta - \cos\theta).$$

Thus the orbit through $(r_0\cos\theta_0, r_0\sin\theta_0)$ *is a circle centered on the line* $u_1 + u_2 = 0$ *and tangent to the line* $u_1 = u_2 = 0$ *at the origin. Since the points of the line* $u_1 - u_2 = 0$ *are all equilibria, we see that all solutions of the 3-homogeneous system which is obtained by setting* $\alpha = 0$ *in* (6.3) *are bounded.*

On the other hand, we observe that $h(\pi/4) = \ell(\pi/4) = 0$*, that* h' *and* ℓ *are positive in the interval* $(\pi/4, 3\pi/4)$*, that* $Q(\pi/4) = -\alpha\epsilon$*, and that* $h' = 2\ell$*, so that* $\nu_* = 1/2$*. Thus Part (a) of Theorem 6.1 shows that if* $\alpha\epsilon < 0$*, the system* (6.3) *has solutions which blow up in a finite time. In particular, we have shown that for any fixed* $\varepsilon > 0$*, all solutions of the system* (6.3) *with* $\alpha = 0$ *are bounded, while if* $\alpha < 0$*, there are solutions which blow up in a finite time. Thus in this case an arbitrarily small linear perturbation from a p-homogeneous system produces solutions with finite-time blowup.*

Mizoguchi, Ninomiya, and Yanagida [1] have shown that the Neumann problem

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= (1-\epsilon)\Delta u_1 + (u_1 - u_2)^3 - u_1, \\
\frac{\partial u_2}{\partial t} &= (1+\epsilon)\Delta u_2 + (u_1 - u_2)^3 - u_2, \\
\frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} = 0 \text{ on } \partial\Omega
\end{aligned} \tag{6.4}$$

on a bounded domain Ω has the property that all solutions converge to the origin when $\epsilon = 0$, but that there are solutions which blow up in a finite time when $0 < \epsilon < 1$.

If we think of $\mathbf{u}(t)$ as the value at $x = 0$ of a functions which is defined to be $\mathbf{0}$ at all the other integer points $x = k \neq 0$, then the the second difference of this grid function at $x = 0$ is $\delta^2\mathbf{u}(t,0) = -2\mathbf{u}(t,0)$. Thus the system in the following example may be thought of as a finite difference analog of the system (6.4) of Mizoguchi, Ninomiya, and Yanagida, but with Dirichlet rather than Neumann boundary conditions.

Example 6.2 *Consider the system*

$$\begin{aligned}
\frac{du_1}{dt} &= (u_1 - u_2)^3 - (3 - 2\epsilon)u_1 \\
\frac{du_2}{dt} &= (u_1 - u_2)^3 - (3 + 2\epsilon)u_2
\end{aligned} \tag{6.5}$$

with

$$0 \leq \epsilon < 1.$$

When $\epsilon = 0$*, we can subtract the second equation of* (6.5) *from the first to see that* $u_1(t) - u_2(t) = [u_1(0) - u_2(0)]e^{-3t}$*. Hence the general solution of* (6.5) *with* $\epsilon = 0$ *is*

$$u_1(t) = u_1(0)e^{-3t} + \frac{1}{6}[u_1(0) - u_2(0)]^3(e^{-3t} - e^{-9t})$$
$$u_2(t) = u_2(0)e^{-3t} + \frac{1}{6}[u_1(0) - u_2(0)]^3(e^{-3t} - e^{-9t}),$$

so that every solution converges to the origin.

The polar form of the system (6.5) *is*

$$\begin{aligned} \frac{d\ln r}{dt} &= r^2(\cos\theta - \sin\theta)^3(\cos\theta + \sin\theta) - 3 + 2\epsilon(\cos^2\theta - \sin^2\theta) \\ \frac{d\theta}{dt} &= (\cos\theta - \sin\theta)^4 - 4\epsilon\sin\theta\cos\theta. \end{aligned} \tag{6.6}$$

We note that $h(\pi/4) = \ell(\pi/4) = 0$*, and that* $\ell(\theta) > 0$ *and* $h'(\theta) < 0$ *in the interval* $(0, \pi/4)$*. Moreover,* $h' = -4\ell$ *so that* $\nu_* = -1/4$*, and* $Q(\pi/4) = 2\epsilon$*. Thus Part (b) of Theorem 6.1 shows that if* $\epsilon > 0$*, no matter how small it may be, there is a solution of* (6.5) *which blows up in finite time.*

The fact that the system (6.5) is a finite difference analog of the version of (6.4) with Neumann conditions replaced by Dirichlet conditions hints, but does not prove, that the result of [1] may be true for Dirichlet boundary conditions as well as for Neumann conditions.

References

[1] N. Mizoguchi, H. Ninomiya, and E. Yanagida (1998). Diffusion-induced blowup in a nonlinear parabolic system, *J. Dynam. Differential Equations* **10**, no. 4, 619–638.

[2] H. Ninomiya and H. F. Weinberger (2003). Pest control may make the pest population explode, *Z. Angew. Math. Phys.* **54**, no. 5, 869–873.