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On optimal extrusion dies for rigid-plastic materials

Hans F. Weinberger

School of Mathematics, University of Minnesota

To my friend George Handelman

1. Introduction. In 1962 Richmond and Devenpeck [2] raised the question of how to design an extrusion die in such a way that the power which must be exerted to convert a bar of a prescribed cross section to a product of prescribed cross section at a prescribed speed is as small as possible. They showed that if one models the problem as a rigid-plastic flow with the Tresca yield condition, this problem can be solved in the case of strip drawing, in which there is no change in a transverse direction. Richmond and Morrison [4] extended this result to wire-drawing, in which case the die and the flow are axially symmetric. In both of these cases, the optimality condition turned out to be that the velocity vector is an eigenvector corresponding to the largest eigenvalue of the rate-of-strain matrix, and the authors called such a flow a streamlined flow.

It was pointed out by Hill [1] that the concept of streamlined flow (which he called ideal flow) can be extended to three dimensions, and gave a heuristic argument to indicate that any optimal die should produce a flow with this property.

In lectures at the Seminar on Industrial Problems of the Institute for Mathematics and its Applications, Owen Richmond and Russ Mallett aroused the author's interest in the problem of computing optimal dies which produce general shapes such as I-beams.

One can approach this problem in two ways. One is to assume that Hill's argument is correct, and to seek a good algorithm to compute a streamlined flow which starts with a prescribed ingot cross-section and ends with a prescribed product cross-section. Some remarks on this approach are to be found in [5], but they did not lead to a satisfactory solution.

The purpose of the present work is to show that while optimization of a two-dimensional or axially symmetric die leads to a streamlined flow, the same is not true for more general dies. Consequently, one must search among a more general class of flows in order to optimize the extrusion of a beam of prescribed cross section.

2. The Tresca variational law. Let $\mathbf{v}(\mathbf{x})$ be a steady velocity field in three-space. The corresponding rate-of-strain matrix is

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

(We use the symbol $_{,i}$ to denote partial differentiation.)

We shall model the flow through the die as a rigid-plastic flow. That is, there is a prescribed closed convex set S in the space of symmetric 3×3 matrices with the property that the stress σ is constrained to lie in this set, and that when σ is in the interior of S, the rate of strain $D = 0$, so that the flow at such points is a rigid body flow.

We shall assume that the remaining flow laws can be obtained from a variational principle in the following manner. For each symmetric matrix D define the support function

$$H(D) = \sup_{\sigma \in S} \mathrm{tr}(\sigma D). \tag{2.1}$$

Then the velocity field is the vector field which minimizes

$$\int H(D) d\mathbf{x} \tag{2.2}$$

where the integral is taken over the material that is being processed (see Figure 1), and

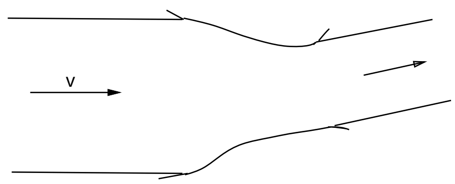

Figure 1

$\mathbf{v}$ is required to be tangential on the die boundary and the lateral boundaries of the material, and to match prescribed constant velocities and cross sections on the upstream and downstream sides. (Because $H(D) = 0$ where the material is rigid, the integration is actually over a finite set.)

The equations of motion are the Euler equations of this variational problem.

We shall consider the Tresca yield criterion, which states that the shear stress on every plane is to be bounded by a positive material constant k. Then

$$S = \{\sigma \ : \ |\mathbf{n}|^2 |\sigma\mathbf{n}|^2 - (\mathbf{n} \cdot \sigma\mathbf{n})^2 \leq k^2 |\mathbf{n}|^4 \text{ for all } \mathbf{n}\}.$$

It is easily checked that in this case

$$H(D) = \begin{cases} 2k||D||_{\ell_2} & \text{when } \mathrm{tr}(D) = 0 \\ +\infty & \text{when } \mathrm{tr}(D) \neq 0 \end{cases},$$

where $||D||_{\ell_2}$ denotes the norm of the linear transformation D on the Euclidean space ℓ_2. The reason for the second line is that if σ lies in S, so does $\sigma - pI$ for all values of p. The effect is that a minimizer of (2.2) must satisfy the constraint $\mathrm{tr}(D) = 0$. That is, $\mathrm{div}(\mathbf{v}) = 0$, so that the flow is incompressible.

The Euler equation for the resulting constrained minimum is most easily written in the following form. The stress is given by

$$\sigma_{ij} = \frac{\partial H(D)}{\partial D_{ij}} - p\delta_{ij} \tag{2.3}$$

where p is a Lagrange multiplier. The equilibrium equations are

$$\sigma_{ij,j} = 0. \tag{2.4}$$

(We shall use the summation convention.) In addition, we have the natural boundary condition

$$\mathbf{n} \times (\sigma \mathbf{n}) = 0$$

on the die and lateral boundaries. This condition states that the boundaries are "well lubricated".

We recall that for the Tresca law $H(D) := 2k||D||_{\ell_2}$, where $||D||_{\ell_2}$ is the norm of the linear transformation D on the Euclidean space ℓ_2. Because D is symmetric, this norm is the the largest of the absolute values of the eigenvalues of D. Let $\mathbf{q}$ be an eigenvector which corresponds to this eigenvalue. We suppose that the determinant of D is not zero. Then the eigenvalue of largest magnitude is uniquely defined, because $\mathrm{tr}(D) = 0$. A computation shows that

$$\frac{\partial ||D||_{\ell_2}}{\partial D_{ij}} = \mathrm{sgn}(\lambda) \frac{q_i q_j}{|\mathbf{q}|^2}. \tag{2.5}$$

We see from (2.3) that the equilibrium conditions (2.4) can be written as

$$(q_{i,j} + q_{j,i})q_j - \frac{1}{|\mathbf{q}|^2} q_k (q_{k,\ell} + q_{\ell,k}) q_\ell \; q_i + q_{k,k} \; q_i = |\mathbf{q}|^2 [\ln |\mathbf{q}| + \mathrm{sgn}(\lambda) p/(2k)]_{,i}.$$

We now choose the otherwise undetermined magnitude

$$|\mathbf{q}| = e^{-\mathrm{sgn}(\lambda)p/(2k)},$$

so that the right-hand side of is zero. Then the equilibrium condition becomes

$$(q_{i,j} + q_{j,i})q_j - \frac{1}{|\mathbf{q}|^2} q_k (q_{k,\ell} + q_{\ell,k}) q_\ell \; q_i + q_{k,k} \; q_i = 0. \tag{2.6}$$

Because the second term on the left is the orthogonal projection of the first term in the direction of $\mathbf{q}$, this system is equivalent to the system

$$\begin{aligned} (q_{i,j} + q_{j,i})q_j - \gamma q_i &= 0 \\ \mathrm{div}\; \mathbf{q} &= 0, \end{aligned}$$

where γ is a new scalar-valued unknown.

The same reasoning shows that the fact that $\mathbf{q}$ is an eigenvector of D and the fact that $\mathrm{div}(\mathbf{v}) = 0$ are equivalent to the system

$$(2.7) \qquad (v_{i,j} + v_{j,i})q_j - \frac{1}{|\mathbf{q}|^2} q_k(v_{k,\ell} + v_{\ell,k})q_\ell \, q_i + v_{k,k} \, q_i = 0.$$

The equations (2.6) and (2.7) are the equations of motion for a rigid- plastic material with the Tresca variational law.

It is easily seen that the system (2.7) is hyperbolic. A surface is characteristic with respect to this system if either it is tangent to $\mathbf{q}$ or its normal makes a 45^o angle with $\mathbf{q}$. Because the system (2.6) is just (2.7) with $\mathbf{v}$ replaced by $\mathbf{q}$, it has the same characteristic surfaces. Therefore the whole system (2.6), (2.7) has these same characteristic surfaces, but each one has multiplicity 2.

3. A sufficient condition for an optimal die. Because the function $H(D) = 2k||D||_{\ell_2}$ is homogeneous of degree one, and because we only work with trace-free matrices D, we see from (2.3) that $\mathrm{tr}(\sigma D) = H(D)$. Thus the integral of $H(D)$, which is minimized for a fixed die, gives the rate of dissipation, which is equal to the power which must be applied in order to force the material through the die. Richmond and Devenpeck defined a die to be optimal if this power is minimized.

In other words, an optimal die is one for which the integral in (2.2) is minimized among all dies which transform a prescribed upstream velocity and cross section to a prescribed downstream velocity and cross section, as well as among all divergence free vector fields $\mathbf{v}$ which satisfy satisfy the upstream and downstream conditions and are tangent to each die boundary.

In order obtain a sufficient condition for optimality, we observe that for any nonzero vector $\mathbf{v}$ and, in particular, for the velocity vector

$$(3.1) \qquad \frac{\mathbf{v} \cdot D\mathbf{v}}{|\mathbf{v}|^2} \leq ||D||_{\ell_2} = H(D)/(2k).$$

We also note that by the definition of D and the fact that $\mathrm{div}(\mathbf{v}) = 0$

$$\frac{\mathbf{v} \cdot D\mathbf{v}}{|\mathbf{v}|^2} = \mathrm{div}\left[(\ln|\mathbf{v}|)\mathbf{v}\right].$$

Thus $H(D)$ is bounded below by a perfect divergence, and the divergence theorem shows that

$$(3.2) \qquad \int H(D)d\mathbf{x} \geq 2kA_{\mathrm{up}}|\mathbf{V}_{\mathrm{up}}| \ln(A_{\mathrm{up}}/A_{\mathrm{down}}).$$

Here A_{up} and A_{down} are the areas of the upstream and downstream cross sections, and $\mathbf{V}_{\mathrm{up}}$ is the prescribed upstream velocity. We have made use of the mass conservation law $A_{\mathrm{up}}|\mathbf{V}_{\mathrm{up}}| = A_{\mathrm{down}}|\mathbf{V}_{\mathrm{down}}|$.

The inequality (3.2) gives a lower bound for the integral to be minimized in terms of the prescribed data only, and is independent of the die shape. Thus if equality holds, the corresponding flow is produced by an optimal die. Equality in (3.2) will hold if and only if equality holds in (3.1) at every point. Because $||D||_{\ell_2}$ is equal to the largest absolute value of the eigenvalues of D, equality in (3.1) is valid if and only if at every point, D has an eigenvalue of largest absolute value which is nonnegative, and $\mathbf{v}$ is a corresponding eigenvector.

Richmond and Devenpeck called a flow with this property a streamlined flow. We have thus shown that if the solution for a given die and given upstream and downstream conditions is a streamlined flow, then the die is optimal.

In terms of the principal eigenvector $\mathbf{q}$, the condition for streamlined flow is just $\mathbf{v} = \beta\mathbf{q}$ for some scalar-valued function β. In general, it seems rather unlikely that the resulting system of 4 unknowns can satisfy the 6 equations (2.6), (2.7). However, because the system (2.6) is just (2.7) with $\mathbf{v}$ replaced by $\mathbf{q}$, one can, indeed satisfy both equations by solving only (2.6) and choosing β to be a constant. This miracle was observed by Richmond and Devenpeck [3] in two dimensions, by Richmond and Morrison [4] for axially symmetric flows, and by Hill [1] in the general case.

Since β is a constant, the equations for streamlined flow are just

$$(3.3) \qquad (v_{i,j} + v_{j,i})v_j - \frac{1}{|\mathbf{v}|^2} v_k(v_{k,\ell} + v_{\ell,k})v_\ell \; v_i + v_{k,k} \; v_i = 0.$$

This system is defined even when $D = 0$ so that $\mathbf{q}$ is undefined.

The system is hyperbolic with simple characteristics. A surface is characteristic if and only if either it is tangent to $\mathbf{v}$ or its normal makes a 45^o angle with $\mathbf{v}$.

4. A necessary condition for optimality. Suppose that the die has the property that it minimizes the integral of $H(D)$, not only among all divergence-free vector fields which satisfy the upstream and downstream conditions, but also among all dies which produce flows which satisfy this condition. We shall use the method of transplantation [2] to obtain a consequence of this fact, which is then a necessary condition for optimality.

Let $\mathbf{y} = \mathbf{y}(\mathbf{x}, \epsilon)$ be a one-parameter family of smooth one-to-one transformations with $\mathbf{y}(\mathbf{x}, 0) = \mathbf{x}$, and define

$$\eta_i(x) = \frac{\partial y_i}{\partial \epsilon}(x, 0).$$

Define the new vector field $\mathbf{V}(\mathbf{y}, \epsilon)$ by

$$V_i(\mathbf{y}(\mathbf{x}, \epsilon), \epsilon) = \frac{1}{\det(\partial \mathbf{y}/\partial \mathbf{x})} v_k(\mathbf{x}) \frac{\partial y_i}{\partial x_k}.$$

A computation shows that

$$\frac{\partial V_i}{\partial y_j} = \frac{1}{\det(\partial \mathbf{y}/\partial \mathbf{x})} \left\{ \frac{\partial}{\partial x_\ell} \left(v_k \frac{\partial y_i}{\partial x_k} \right) - v_k \frac{\partial y_i}{\partial x_k} \frac{\partial^2 y_n}{\partial x_\ell \partial x_m} \frac{\partial x_m}{\partial y_n} \right\} \frac{\partial x_\ell}{\partial y_j}.$$

In particular, we find by putting $j = i$ and summing that

$$\sum \frac{\partial V_i}{\partial y_i} = \frac{1}{\det(\partial \mathbf{y}/\partial \mathbf{x})} \sum \frac{\partial v_i}{\partial x_i}.$$

Since $\mathbf{v}$ is divergence-free, the vector field $\mathbf{V}$ is divergence free on the image of the domain under the mapping.

Let $D_{ij}(\mathbf{y}, \epsilon) = (1/2)(\partial V_i/\partial y_j + \partial V_i/\partial y_j)$. We note that

$$\int_{y(\Omega,\epsilon)} H(D(y,\epsilon)dy = \int_\Omega H(D(y(x,\epsilon),\epsilon)\det(\partial \mathbf{y}/\partial \mathbf{x}))dx.$$

It is obvious from the definitions of $\mathbf{y}(\mathbf{x}, \epsilon)$ and of $\mathbf{V}$ that at $\epsilon = 0$ $\mathbf{V}(\mathbf{y}(\mathbf{x}, 0)), 0) = \mathbf{v}(\mathbf{x})$, and hence also $\partial V_i/\partial y_j = \partial v_i/\partial x_j$. Moreover, $\det(\partial \mathbf{y}/\partial \mathbf{x})$=1, and $\partial x_i/\partial \epsilon = -\eta_i(x)$ at $\epsilon = 0$. Therefore

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}\left[\frac{\partial V_i}{\partial y_j}\det(\partial y/\partial x)\right]_{\epsilon=0} &= \frac{\partial}{\partial x_j}\left(v_k \frac{\partial \eta_i}{\partial x_k}\right) - \frac{\partial v_i}{\partial x_k}\frac{\partial \eta_k}{\partial x_j} - v_i \frac{\partial^2 \eta_k}{\partial x_j \partial x_k} \\
&= \frac{\partial}{\partial x_j}\left(v_k \frac{\partial \eta_i}{\partial x_k} - \frac{\partial v_i}{\partial x_k}\eta_k - v_i \frac{\partial \eta_k}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_j}\eta_k\right).
\end{aligned}$$

Thus we see from the divergence theorem and the equilibrium equation that

(4.1)

$$\begin{aligned}
\frac{d}{d\epsilon}\left[\int H(D)dy\right]_{\epsilon=0} &= \int \frac{\partial H}{\partial D_{ij}}\frac{\partial}{\partial \epsilon}\left(\det(\partial \mathbf{y}/\partial \mathbf{x})\frac{\partial V_i}{\partial y_j}\right)dx \\
&= \int \sigma_{ij}\frac{\partial}{\partial \epsilon}\left(\det(\partial \mathbf{y}/\partial \mathbf{x})\frac{\partial V_i}{\partial y_j}\right)dx \\
&= \int_{bdy}\left\{\sigma_{ij}n_j\left(v_k \frac{\partial \eta_i}{\partial x_k} - \frac{\partial v_i}{\partial x_k}\eta_k - v_i \frac{\partial \eta_k}{\partial x_k}\right) + \sigma_{ij}\frac{\partial v_i}{\partial x_j}\eta_k n_k\right\}dS \\
&\qquad - \int \eta_k \frac{\partial v_i}{\partial x_j}\frac{\partial \sigma_{ij}}{\partial x_k}dx.
\end{aligned}$$

Because $S_{ij} := \partial H/\partial D_{ij}$ is homogeneous of degree zero and div $\mathbf{v}$=0, we see that

$$\frac{\partial v_i}{\partial x_j}\frac{\partial \sigma_{ij}}{\partial x_k} = D_{ij}\frac{\partial S_{ij}}{\partial D_{\ell m}}\frac{\partial D_{\ell m}}{\partial x_k} = 0,$$

so that the last integral in (4.1) is zero.

We now look at the boundary integral in the next-to-the-last line. We shall assume that η vanishes outside the domain where nonrigid flow occurs, so that this integral is over the lateral boundary. Here we know that $\sigma_{ij}n_j = \alpha n_i$ for some scalar-valued function α, and that $\mathbf{v} \cdot \mathbf{n} = 0$.

We observe that

$$v_k \frac{\partial \eta_i}{\partial x_k} - \frac{\partial v_i}{\partial x_k}\eta_k - v_i \frac{\partial \eta_k}{\partial x_k} = [\mathrm{curl}(\eta \times \mathbf{v})]_i.$$

Thus the first term in the boundary integral becomes

$$\begin{aligned}\int_{bdy} \alpha\mathbf{n}\cdot \mathrm{curl}(\eta\times\mathbf{v})\ dS &= \int_{bdy} [\mathbf{n}\cdot\mathrm{curl}(\alpha\eta\times\mathbf{v}) - \mathbf{n}\cdot\nabla\alpha\times(\eta\times\mathbf{v})]dS \\ &= -\int_{bdy} \mathbf{v}\cdot\nabla\alpha\ \eta\cdot\mathbf{n}\ dS\end{aligned}$$

by the Stokes theorem.

Because of the equilibrium equation (2.4), $\mathrm{div}(\sigma\mathbf{v}) = \sigma_{ij}\partial v_i/\partial x_j$. Thus (4.1) becomes

$$\frac{d}{d\epsilon}\left[\int H(D)dy\right]_{\epsilon=0} = \int_{bdy} \mathrm{div}[-\alpha\mathbf{v}+\sigma\mathbf{v}]\eta\cdot\mathbf{n}\ dS. \tag{4.2}$$

If the domain minimizes the dissipation, this integral must be zero for arbitrary boundary values of η and we conclude that we must have

$$\mathrm{div}[-\alpha\mathbf{v}+\sigma\mathbf{v}] = 0.$$

We see from the formula (2.3) for σ and the formula (2.5) that $\sigma\mathbf{n} = 2k\ \mathrm{sgn}(\lambda)(\mathbf{q}\cdot\mathbf{n}/|\mathbf{q}|^2)\mathbf{q} - p\mathbf{n}$. Then the well-lubricated boundary condition shows that $\mathbf{q}$ is either normal or tangent to the boundary. In the former case, $\alpha = 2k\mathrm{sgn}(\lambda) - p$, while in the latter case $\alpha = -p$. In either case, we see that because $\mathrm{div}(\mathbf{v}) = 0$, $\mathrm{div}[-\alpha\mathbf{v}] = \mathrm{div}[p\mathbf{v}]$.

Thus in view of (2.3) and (2.5) the necessary condition becomes

$$\mathrm{div}\left[\frac{\mathbf{v}\cdot\mathbf{q}}{|\mathbf{q}|^2}\mathbf{q}\right] = 0 \tag{4.3}$$

on the boundary of an optimal die.

5. Two–dimensional and axially symmetric optimal dies. We now look at the special case of plane flow. That is, we suppose that $v_3 = 0$, and that the other two components are independent of x^3. In this case, D has a zero eigenvalue and the other two eigenvalues are negatives of each other. Since we deal only with matrices which satisfy the constraints, the differentiation formula (2.5) is still valid, modulo a scalar matrix. Because the two eigenvalues have equal magnitude, we may take the principal eigenvector $\mathbf{q}$ to be in the tangent direction. In this case, $\mathbf{v} = \beta\mathbf{q}$ on the boundary, and, since div $\mathbf{q} = \mathbf{0}$, the necessary condition (4.3) on the boundary becomes $\mathbf{q}\cdot\nabla\beta = 0$. Thus β is a constant.

The left-hand sides of the equations (2.6) and (2.7) with $i = 3$ are just zero. By subtracting β times (2.6) from (2.7), we obtain the system

$$\begin{aligned}&\begin{pmatrix}(q_1^2+3q_2^2)q_1 & (q_2^2-q_1^2)q_2 \\ (q_2^2-q_1^2)q_2 & (q_1^2-q_2^2)q_1\end{pmatrix}\frac{\partial}{\partial x_1}\begin{pmatrix}v_1-\beta q_1 \\ v_2-\beta q_2\end{pmatrix} \\ &\qquad + \begin{pmatrix}(q_2^2-q_1^2)q_2 & (q_1^2-q_2^2)q_1 \\ (q_1^2-q_2^2)q_1 & (3q_1^2+q_2^2)q_2\end{pmatrix}\frac{\partial}{\partial x_2}\begin{pmatrix}v_1-\beta q_1 \\ v_2-\beta q_2\end{pmatrix} = 0.\end{aligned} \tag{5.1}$$

This is easily seen to be a hyperbolic system whose characteristics make the angles $\pm\pi/4$ with the direction of $\mathbf{q}$.

Since $\mathbf{q}$ is tangent to the die boundary, this boundary is not characteristic. Because the vector field $\mathbf{v} - \beta\mathbf{q}$ is a solution of the linear system (5.1) and vanishes on a noncharacteristic curve, it vanishes throughout the characteristic triangles determined by the die boundary. Thus the necessary condition (4.3) implies that $\mathbf{v}$ is an eigenvector of D in these triangles. If λ is the eigenvalue whose eigenvector is $\mathbf{v}$, then $\lambda = \mathbf{v} \cdot D\mathbf{v}/|\mathbf{v}|^2 = \mathbf{v} \cdot \nabla \ln |\mathbf{v}|$. Since the speed is larger downstream than upstream, we must have $\lambda > 0$. Thus, the necessary condition (4.3) implies that the flow is streamlined in the characteristic triangles determined by the upper and lower die boundaries. Because the flow is rigid outside these triangles, it is streamlined everywhere.

Richmond and Devenpeck [3] have, in fact, shown how to construct optimal two-dimensional dies in the following manner. In two dimensions the equations (3.3) of streamlined flow reduce to

(5.2)

$$\begin{pmatrix} (v_1^2 + 3v_2^2)v_1 & (v_2^2 - v_1^2)v_2 \\ (v_2^2 - v_1^2)v_2 & (v_1^2 - v_2^2)v_1 \end{pmatrix} \frac{\partial}{\partial x_1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} (v_2^2 - v_1^2)v_2 & (v_1^2 - v_2^2)v_1 \\ (v_1^2 - v_2^2)v_1 & (3v_1^2 + v_2^2)v_2 \end{pmatrix} \frac{\partial}{\partial x_2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$

which is a hyperbolic system whose characteristics are at 45^o to $\mathbf{v}$.

Prescribe the restriction $\mathbf{v}(x^1, 0)$ of $\mathbf{v}$ along the line $x_2 = 0$, with $v_2(x^1, 0) = 0$ so that the line is a streamline, and with $v_1(x^1, 0)$ a continuously differentiable function which is equal to the given inflow speed for $x_1 \leq c$ and to the given outflow speed for $x_1 \geq d$, and whose derivative is positive for $c < x_1 < d$. Solve the system (5.2) both above and below the line with the prescribed initial values of $\mathbf{v}$ at $x_2 = 0$.

It is easy to see that the solution $\mathbf{v}$ of this initial value problem is equal to the prescribed constant horizontal upstream velocity to the left of the domain of influence of the interval (c, d) of the x^1-axis, and to the downstream velocity to the right of this domain of influence.

One thus obtains the velocity field $\mathbf{v}$ of a streamlined flow. Any pair of streamlines of this flow gives an optimal die which converts an upstream flow with the prescribed speed and width to a downstream flow of the prescribed speed and the corresponding width.

Similar considerations apply for an axially symmetric flow. Here the velocity is constrained to be of the form

$$\mathbf{v} = \begin{pmatrix} u(r,z)\cos\theta \\ u(r,z)\sin\theta \\ w(r,z) \end{pmatrix}$$

so that

$$D = \begin{pmatrix} \frac{\partial u}{\partial r}\cos^2\theta + \frac{u}{r}\sin^2\theta & \left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\sin\theta\cos\theta & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\cos\theta \\ \left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\sin\theta\cos\theta & \frac{\partial u}{\partial r}\sin^2\theta + \frac{u}{r}\cos^2\theta & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\sin\theta \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\cos\theta & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\sin\theta & \frac{\partial w}{\partial z} \end{pmatrix}.$$

This matrix has the eigenvalue u/r with the eigenvector $(-\sin\theta,\ \cos\theta)$, which is perpendicular to $\mathbf{v}$. In addition, this matrix has two other eigenvalues, whose eigenvectors are of the same form as $\mathbf{v}$. Unless $u = 0$, the other two eigenvalues are not negatives of each other.

If the eigenvector $\mathbf{q}$ which corresponds to the eigenvalue of largest magnitude is of the same form as $\mathbf{v}$ and tangent to the die boundary, then one can carry through the same argument as in the two-dimensional case to show that the necessary condition (4.3) implies that the flow is streamlined, so that the die is optimal.

In order to make these rather strong assumptions on $\mathbf{q}$ more plausible, we note that if the flow is smooth near the z-axis, then on the z-axis $D =$diag$(\partial u/\partial r,\ \partial u/\partial r,\ \partial w/\partial z)$. We see from the divergence condition that the largest eigenvalue is $\partial w/\partial z = -2\partial u/\partial r$. Thus on the z-axis $\mathbf{q}$ is in the z-direction. Moreover, because the velocity is larger downstream than upstream, $\lambda = \partial w/\partial z > 0$ somewhere on the z-axis, and $u/r < 0$ near this point. Thus if we onlyn assume that D never has the eigenvalue zero in the flow region, we conclude that $\lambda > 0$ and $u < 0$ everywhere. The latter condition states that the die is constricting.

Suppose now that the die boundary is connected, and that $\mathbf{q}$ is normal to the die boundary. Then $\mathbf{v}$ is an eigenvector of D, and because $\lambda > 0$, the eigenvalue corresponding to v must be negative. It follows that $\mathbf{v} \cdot \nabla |\mathbf{v}|^2 = 2\mathbf{v} \cdot D\mathbf{v} < 0$. This contradicts the fact that the downstream speed is greater than the upstream speed. Consequently, $\mathbf{q}$ must be tangent to the boundary, and the analogue of the two-dimensional proof shows thast the flow is streamlined. We have thus shown that if an axially symmetric die is optimal and if the determinant of D does not vanish in the nonrigid flow region, then the flow is streamlined.

Richmond and Morrison [4] have shown how to construct a streamlined axially symmetric flow by solving the system of equations for (u, w) with prescribed initial conditions on the axis of symmetry in a manner similar to the two-dimensional case. This again gives a flow whose streamlines can serve as optimal die boundaries.

6. The general die problem. In the general three-dimensional case the necessary condition (4.3) is far from implying that the flow is streamlined.

If the eigenvector $\mathbf{q}$ which corresponds to the maximal eigenvalue of D is normal to the boundary, the flow cannot be streamlined. Even if we suppose that $\mathbf{q}$ is tangent to the boundary, the necessary condition (4.3) only implies that there is a scalar-valued function β which is constant on $\mathbf{q}$-streamlines of the boundary such that the vector field $\mathbf{v} - \beta\mathbf{q}$ is perpendicular to $\mathbf{q}$ on the die boundary. This does not lead to the conclusion that $\mathbf{v}$ has the same direction as $\mathbf{q}$. Thus the necessary condition (4.3) does not seem to imply that the flow must be streamlined.

One could still hope to apply a construction to like that of Richmond and Devenpeck in which one specifies a tangent vector field $\mathbf{v}$ on a surface which cuts the material lengthwise, with the prescribed constant velocities upstream and downstream, and then determines the flow from the streamlined flow equations (3.3).

The system (3.3) is hyperbolic, and its normal cone consists of the directions which are perpendicular to $\mathbf{v}$ and those which make a 45^o angle with $\mathbf{v}$. Therefore the above initial surface is both characteristic and timelike. The latter means that the Cauchy problem with such an initial surface is not well-posed. That is, it is not practical to try to compute a streamlined flow from data on such a surface. Moreover, even if one could compute such a flow, there is no reason to believe that the solution of the system (3.3) whose values on the initial surface have the prescribed constant velocities on the upstream and downstream parts of this surface will have these same constant velocities in any region off the surface, or that the prescribed cross-sections are stream surfaces of this flow.

More generally, we observe that the upstream and downstream shapes of a die for two-dimensional or axially symmetric flow are specified by a small number of parameters: The upstream and downstream velocities and the upstream width or radius. By contrast, in a general extrusion problem one must specify the upstream and downstream cross-sectional shapes, each of which involves infinitely many parameters. NOTE ADDED IN PROOF: The author has, in fact, shown [J. Mech. Phys. Solids **45**, 1275-1280 (1997) that there are pairs of upstream and downstream cross sections for which there is no die which produces a streamlined flow.

These considerations make it quite unlikely that constructing special streamlined flows is useful in trying to optimize a general die.

On the other hand, the variational formula (4.2) indicates how to make small adjustments in the die boundary which decrease the rate of dissipation. One must, of course, take care to preserve the downstream cross section in this process. Such considerations may lead to a useful algorithm for approaching the optimal shape of a die.

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