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ON SUFFICIENT CONDITIONS FOR A LINEARLY DETERMINATE SPREADING SPEED

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To Avner Friedman on the occasion of his 80th year of youthful enthusiasm.

Abstract. It is shown how to construct criteria of the form $f(u) \leq f'(0)K(u)$ which guarantee that the spreading speed c^* of a reaction-diffusion equation with the reaction term $f(u)$ is linearly determinate in the sense that $c^* = 2\sqrt{f'(0)}$. Some of these criteria improve the classical condition $f(u) \leq f'(0)u$, and permit the presence of sharp Allee effects. Inequalities which guarantee the failure of linear determinacy are also presented.

1. **Introduction.** We consider the asymptotic spreading speed c^* of the reaction-diffusion equation

$$u_t = u_{xx} + f(u) \tag{1.1}$$

in the monostable case. That is, we assume that $f(0) = f(1) = 0$ and $f(u) > 0$ for $0 < u < 1$. Linearization about zero immediately shows that $c^* \geq 2\sqrt{f'(0)}$. The spreading speed is said to be **linearly determinate** (or, equivalently, the linear conjecture is said to be valid) when $c^* = 2\sqrt{f'(0)}$. It is well known that the classical condition

$$f(u) \leq f'(0)u \tag{1.2}$$

implies the linear determinacy of the spreading speed of (1.1). This condition is less restrictive than the condition $f'(u) \leq f'(0)$ introduced by Kolmogorov, Petrovski and Piscounov [4] or the condition that the per capita net growth rate $f(u)/u$ is nonincreasing. Because it is difficult to calculate c^* when it is not equal to $2\sqrt{f'(0)}$, it is useful to have other conditions of the form

$$f(u) \leq f'(0)K(u) \tag{1.3}$$

which imply that (1.1) is linearly determinate

This work gives a systematic method for finding functions $K(u)$ with $K'(0) = 1$ for which the inequality (1.3) implies that (1.1) is linearly determinate. For instance, Example 2.2 with $\gamma = 1/2$ shows that the function

$$K(u) = 2\left[1 - \sqrt{1-u}\right] \tag{1.4}$$

has this property. Since $K(u) = 2u/[1 + \sqrt{1-u}] > u$ for $0 < u \leq 1$, the sufficient condition (1.3) is an improvement of the classical condition (1.2).

Key words and phrases. reaction-diffusion systems, spreading speed.

Section 2 presents the basic method for finding functions $K(u)$ with the property that the inequality (1.3) implies that the spreading speed of (1.1) is linearly determinate. The idea is to require an upper bound for c^* due to Hadeler and Rothe [3] to be equal to the lower bound $2\sqrt{f'(0)}$.

In Section 3 we present a method of constructing a complementary lower bound for f which implies that the spreading speed of (1.1) is not linearly determinate. The method here comes from another theorem of Hadeler and Rothe [3].

The Allee effect is defined as the property that the per capita net growth rate $f(u)/u$ is increasing when the population density u is sufficiently low. It was proposed by Allee [1] as one possible cause of the spacial aggregation of a population in a homogeneous environment, or of the segregation of similar species. While Allee did not discuss the question of spreading speeds, the Allee effect is sometimes thought to be related to the property of linear determinacy.

Wang and Kot [5], gave examples to show that an $f(u)$ with $f(u)/u$ neither increasing nor decreasing for small u may or may not produce linear determinacy,

Because the function K defined by (1.4) has the property that $K(u)/u$ is increasing, the inequality $f(u) \leq K(u)$ which implies linear determinacy can be satisfied by a function f with a pronounced Allee effect. In fact, we shall construct other examples of K which permit even sharper Allee effects without harming the linear determinacy.

Similarly, the results of Section 3 produce examples in which $f(u)/u$ is strictly decreasing for small u, but the spreading speed is greater than the linear speed $2\sqrt{f'(0)}$. That is, the absence of an Allee effect does not prevent the failure of linear determinacy.

Section 4 shows that an idea of Gilding and Kersner [2] can be used to extend our results to density-dependent reaction-advection-diffusion models.

2. **A sufficient condition for linear determinacy.**

We shall consider the spreading speed of the reaction-diffusion diffusion equation (1.1) by looking at its traveling wave solutions $w(x-ct)$. Such a solution satisfies the ordinary differential equation

$$w'' + cw' + f(w) = 0. \tag{2.1}$$

We shall assume that f is continuous in $[0,1]$, that it is right differentiable at 0, and that

$$f(0) = f(1) = 0, f'(0) > 0, \text{ and } f(u) > 0 \text{ for } 0 < u < 1. \tag{2.2}$$

Let $u(x,t)$ be a solution of (1.1) such that $u(x,0) \not\equiv 0$ has values in [0,1) and vanishes outside a bounded interval. It is known that if, for a fixed value $\bar{u}$ between 0 and 1, one defines $x_{\bar{u}}(t)$ to be the largest value of x at which $u(x,t) = \bar{u}$, then the ratio $x_{\bar{u}}(t)/t$ has a limit c^* as $t \to \infty$ which is independent of $\bar{u}$. c^* is called the **rightward asymptotic spreading speed**. It is also known that c^* can be characterized as the smallest value of c for which there is a nonincreasing solution w of (2.1) with

$$w(-\infty) = 1 \text{ and } w(\infty) = 0.$$

It is easily seen that when $c < 2\sqrt{f'(0)}$, every solution of (2.1) which approaches zero must oscillate about zero, so that it cannot be nonnegative. Therefore, the speed c of any nonnegative traveling wave must satisfy the inequality $c \geq 2\sqrt{f'(0)}$, which gives the lower bound

$$c^* \geq 2\sqrt{f'(0)}, \tag{2.3}$$

Hadeler and Rothe [3] used phase plane analysis to obtain an upper bound for the spreading speed of the reaction-diffusion equation (1.1). Their Theorem 8 is equivalent to the following proposition.

Proposition 1. *Suppose that there are a function $\phi(u)$ which is continuous in $[0,1]$ and differentiable in $[0,1)$ and a number c with the following properties.*

$$\begin{aligned} &a.\ \phi(0)=0,\\ &b.\ \phi(u)>0 \text{ in } (0,1), \text{ and}\\ &c.\ \phi(u)\phi'(u)-c\phi(u)+f(u)\le 0 \text{ in } (0,1). \end{aligned} \tag{2.4}$$

Then the spreading speed of the equation (1.1) satisfies the inequality

$$c^* \le c. \tag{2.5}$$

Remark. In order to write the result in this form, we have replaced the function $\rho(u)$ in [3] by the function $\phi(1-u)$.

The right-hand side of (2.3) depends on $f(u)$ only through its linearization at 0. For this reason, the spreading speed of (1) is said to be **linearly determinate** if it happens that $c^*=2\sqrt{f'(0)}$. In view of the lower bound (2.3), the property that c^* is bounded above by $2\sqrt{f'(0)}$ implies that c^* is linearly determinate. While the idea of Hadeler and Rothe is to prescribe f and ϕ and search for a c such that the inequality (2.4.c) is satisfied, we shall prescribe ϕ and c, and consider (2.4.c) as a condition to be satisfied by f.

In order to implement this idea, we first observe that because ϕ and f are positive in $(0,1)$, the inequality (2.4.c) implies that $\phi'(u)<c$ in $(0,1)$. Because the left-hand side of (2.4.c) vanishes at 0, its derivative at 0 is nonpositive. That is,

$$\phi'(0)^2 - c\phi'(0) + f'(0) \le 0.$$

We now let $c=2\sqrt{f'(0)}$. Then the left-hand side of this inequality is $[\phi'(0)-\sqrt{f'(0)}]^2$, so that $\phi'(0)=\sqrt{f'(0)}$. Because $f>0$ in $(0,1)$, (2.4.c) shows that $c=2\phi'(0)>\phi'(u)$ in $(0,1)$. Thus, when $c=2\sqrt{f'(0)}$, ϕ must satisfy the conditions

a. ϕ is continuous on the interval $[0,1]$ and continuously differentiable in [0,1);
b. $\phi(0)=0,\ \phi'(0)>0,\ \phi(u)>0$ in $(0,1)$; and
c. $\phi'(u)<2\phi'(0)$ in $[0,1)$. (2.6)

Our basic result is the following theorem

Theorem 2.1. *Let $\phi(u)$ be any function which is continuous on $[0,1]$ and continuously differentiable in [0,1), and which has the properties (2.6). Define the function*

$$K_\phi(u) := \phi(u)[2\phi'(0)-\phi'(u)]/\phi'(0)^2. \tag{2.7}$$

Let $f(u)$ satisfy the conditions (2.2). If f also satisfies the inequality

$$f(u) \le f'(0)K_\phi(u), \tag{2.8}$$

then the spreading speed c^* of the equation (1.1) is linearly determinate. That is,

$$c^* = 2\sqrt{f'(0)}. \tag{2.9}$$

Proof. We observe that if ϕ has the properties (2.6), then the same is true of any positive multiple of ϕ. For a fixed function f with the stated properties, we define the function
$$\hat{\phi}(u) = \sqrt{f'(0)}\phi(u)/\phi'(0).$$
Then the defining equation (2.7) can be written in the form
$$\hat{\phi}(u)\hat{\phi}'(u) - 2\sqrt{f'(0)}\hat{\phi}(u) + f'(0)K_{\phi}(u) = 0.$$
Thus if $f(u) \leq f'(0)K_{\phi}(u)$, the conditions (2.4) with $c = 2\sqrt{f'(0)}$ are satisfied by $\hat{\phi}$. Then the Hadeler-Rothe bound shows that $c^* \leq 2\sqrt{\mathbf{f'(0)}}$. We combine this with the lower bound ((2.3) to see that $c^* = 2\sqrt{\mathbf{f'(0)}}$. This is the statement of the Theorem. □

Remark. The definition (2.7) shows that $K'_{\phi}(0) = 1$, so that the function $f'(0)K_{\phi}$ has the derivative $f'(0)$ at $u = 0$.

The simplest choice of the function ϕ is the linear function $\phi(u) = u$. In this case, $K_{\phi}(u) = u$, so that the condition (2.8) is the classical condition $f(u) \leq f'(0)u$. The following example embeds this choice in a two-parameter family.

Example 2.1. We choose the family
$$\phi(u) = u(1 - \gamma u^{\delta})$$
with
$$0 < \gamma \leq 1 \text{ and } \delta > 0.$$
Then the condition (2.8) becomes
$$f(u) \leq f'(0)u[1 + \gamma\delta u^{\delta} - \gamma^2(\delta + 1)u^{2\delta}]. \tag{2.10}$$
When
$$\gamma \leq \delta/(\delta + 1).$$
the right-hand side of (2.10) is larger than u in $(0, 1)$, so that the inequality (2.10) improves the classical inequality $f(u) \leq u$. If, as in Theorem 2.1, we denote the coefficient of $f'(0)$ on the right-hand side of (2.10) by K_{ϕ}, we see that
$$[K_{\phi}(u)/u]' = 2\gamma^2\delta(\delta + 1)u^{\delta-1}\left[\frac{\delta}{2\gamma(\delta + 1)} - u^{\delta}\right]$$
Thus, if the inequalites
$$0 < \delta < 1 \text{ and } \gamma \leq \delta/[2(\delta + 1)]$$
are satisfied, $K_{\phi}(u)/u$ is increasing on the interval [0,1], and its derivative approaches infinity at zero. By defining $f(u) = f'(0)K_{\phi}(u)$ with the exception of a small neighborhood of $u = 1$ where f decreases to 0, one sees that even when f has a very sharp Allee effect in a very large neighborhood of $u = 0$, the spreading speed of (1.1) can be linearly determinate. □

Remark. The case $\gamma = \delta = 1$ of the above example was used by Hadeler and Rothe [3] to show that when $-l \leq \nu \leq 2$, the equation (1.1) with $f(u) = u(1 - u)(1 + \nu u)$ has the linearly determinate spreading speed 2.

The following definition is useful for analyzing the result of Theorem 2.1.

Definition 2.2. The sufficient condition
$$f(u) \leq f'(0)K_{\phi}(u) \tag{2.11}$$

for the spreading speed of (1.1) to be linearly determinate is said to be **improvable** if there is a function $\tilde{\phi}$ which satisfies the conditions of Theorem 2.1 and the additional condition

$$\begin{aligned} K_{\tilde{\phi}}(u) &:= \tilde{\phi}(u)[2\tilde{\phi}'(0) - \tilde{\phi}'(u)]/\tilde{\phi}'(0)^2 \\ &\geq K_\phi := \phi(u)[2\phi'(0) - \phi'(u)]/\phi'(0)^2 \text{in } (0,1), \end{aligned}$$

with strict equality at at least one u in $(0,1)$.

If there is no $\tilde{\phi}$ with this property, the condition $f(u) \leq \mathbf{f}'(0)K_\phi(u)$ is said to be **extremal**.

Remark. By definition, if the conditions $f(u) \leq f'(0)K_\phi(u)$ and $f(u) \leq K_{\tilde{\phi}}(u)$ are both extremal and $K_{\tilde{\phi}} \not\equiv K_\phi$, then there must be points where $K_{\tilde{\phi}} > K_\phi$ and other points where $K_{\tilde{\phi}} < K_\phi$.

We note that if (2.11) is improvable, the sufficient condition $f(u) \leq f'(0)K_\phi(u)$ can be improved by replacing it with the less restrictive condition $f \leq f'(0)K_{\tilde{\phi}}(u)$. No such improvement is possible if $\hat{f}$ is extremal. Thus, it is useful to characterize the extremal functions $\hat{f}$

Theorem 2.3. *The sufficient condition*

$$f(u) \leq f'(0)K_\phi(u)$$

where ϕ satisfies the conditions of Theorem 2.1 is extremal if and only if ϕ satisfies the additional conditions

$$\phi(1) = 0 \tag{2.12}$$

and

$$\liminf_{u \searrow 0} \int_u^{1/2} [\{\phi'(0) - \phi'(v)\}/\phi(v)]dv < +\infty. \tag{2.13}$$

Proof. Suppose, for the sake of contradiction, that the conditions (2.12) and (2.13) are satisfied, but that the inequality $f \leq f'(0)K_\phi(u)$ is improvable. That is, there is a function $\tilde{\phi}$ which has the properties (2.6,) such that $K_{\tilde{\phi}} \geq K_\phi$ with strict inequality somewhere. Because the expression for K_ϕ in (2.7) and the integral in (2.13) are homogeneous of degree zero in ϕ, we may assume without loss of generality that $\tilde{\phi}'(0) = \phi'(0)$. Then the inequality $K_{\tilde{\phi}} \geq K_\phi$ takes the form

$$\tilde{\phi}(u)[2\phi'(0) - \tilde{\phi}'(u)] \geq \phi(u)[2\phi'(0) - \phi'(u)], \tag{2.14}$$

with strict inequality on some open interval. This inequality can be written in the form $[\tilde{\phi}^2 - \phi^2]' \leq 4\phi'(0)[\tilde{\phi} + \phi]^{-1}[\tilde{\phi}^2 - \phi^2]$, or equivalently as

$$\left\{ e^{4\phi'(0)\int_u^{1/2}[\tilde{\phi}(v)+\phi(v)]^{-1}dv}(\tilde{\phi}^2 - \phi^2) \right\}' \leq 0. \tag{2.15}$$

That is, the function in braces is non-increasing.

We first examine the function in braces near $u = 1$. Since the integral is non-positive for $u \geq 1/2$ and $\phi(1) = 0$, we see that its limit inferior at 1 is nonnegative. Then (2.15) shows that $\tilde{\phi}^2 - \phi^2 \geq 0$ in $(0,1)$. Because ϕ and $\tilde{\phi}$ are positive, we have $\tilde{\phi} \geq \phi$ in this interval. Therefore, $4/[\tilde{\phi}(v) + \phi(v)] \leq 2/\phi(v)$. Thus, the assumption (2.13) implies that there is a sequence u_i which converges to 0 such that the the exponential function in (2.15) at u_i is bounded above by a multiple of $[\phi(u_i)]^{-2}$. Since $\phi(0) = \tilde{\phi}(0) = 0$ and $\tilde{\phi}'(0) = \phi'(0)$, the nonnegative function $[\phi(u_i)]^{-2}[\tilde{\phi}(u_i)^2 - \phi(u_i)^2]$ converges to 0. Thus the nonincreasing nonnegative function in the braces of (2.15) converges to 0 at $u = 0$. Consequently, it is identically zero. We have shown that, under the conditions of the Theorem, the inequality $K_{\tilde{\phi}} \geq K_\phi$ implies that $\tilde{\phi} \equiv \phi$, so that $K_{\tilde{\phi}} \equiv K_\phi$. By definition, this means that the condition $f(u) \leq f'(0)K_\phi(u)$ is extremal.

To prove the necessity of the condition $\phi(1) = 0$, we suppose that $\phi(1) > 0$. Then there is a positive constant m such that

$$\phi(u) \geq 2m \text{ for } 1/2 \leq u \leq 1.$$

We define the function

$$\tilde{\phi}(u) := \phi(u) - \epsilon\eta(u),$$

where

$$\eta(u) = \begin{cases} 0 & \text{for } u \leq 1/2 \\ [u-(1/2)]^2 e^{\{2\phi'(0)u - \phi(u)\}/m} & \text{for } 1/2 \leq u \leq 1, \end{cases}$$

and ϵ is a small positive constant. The formula (2.7) shows that

$$\phi'(0)^2[K_{\tilde{\phi}}(u) - K_\phi(u)] = -\epsilon[2\phi'(0) - \phi'(u)]\eta(u) + \epsilon[\phi(u) - \epsilon\eta(u)]\eta'(u).$$

By using the formula for η, we see that the right-hand side is zero for $u \leq 1/2$, and that this right-hand side is given by

$$\begin{aligned} &\epsilon\eta(u)\big[[2\phi'(0) - \phi'(u)]\{-1 + m^{-1}[\phi(u) - \epsilon\eta(u)]\} \\ &\quad + 2\epsilon[\phi(u) - \epsilon\eta(u)][u-(1/2)][\phi'(0)]^{-2}e^{\{2\phi'(0) - \phi(u)\}/m}\big] \end{aligned} \tag{2.16}$$

for $u \geq 1/2$. Because $\phi(u) \geq 2m$ and η is bounded, we can choose a positive ϵ which is so small that

$$\phi(u) - \epsilon\eta(u) \geq m \text{ for } 1/2 \leq u \leq 1.$$

Then the factor in braces in (2.16) is nonnegative. Thus we find that $K_{\tilde{\phi}}(u) - K_\phi(u)$, which is zero for $u \leq 1/2$, is bounded below by a positive multiple of $[u-(1/2)]$ for $u > 1/2$. We conclude that if $u(1) > 0$, then the condition $f(u) \leq f'(0)K_\phi(u)$ is improvable rather that extremal.

To show that the condition (2.13) is also necessary, we suppose that it is not satisfied by ϕ. That is, the integral on the left of (2.13) approaches infinity as u goes to 0. We define

$$\tilde{\phi}(u) = \phi(u) + \epsilon\zeta(u)$$

where $\epsilon > 0$ and

$$\zeta(u) := \begin{cases} [(1/2) - u]^2 \phi(u) e^{-2\int_u^{1/2}\{[\phi'(0) - \phi'(v)]/\phi(v)\}dv} & \text{for } 0 \leq u \leq 1/2 \\ 0 & \text{for } 1/2 \leq u \leq 1. \end{cases} \tag{2.17}$$

Because the integral in the exponential goes to infinity as u decreases to 0, $\zeta(u)/u$ approaches zero at 0. Therefore,

$$\zeta(0) = \zeta'(0) = 0.$$

It follows that

$$\tilde{\phi}(0) = 0 \text{ and } \tilde{\phi}'(0) = \phi'(0).$$

A calculation shows that for $u \leq 1/2$

$$\begin{aligned} &\phi'(0)^2[K_{\tilde{\phi}}(u) - K_\phi(u)] = \epsilon[(1/2) - u]\phi(u)e^{-2\int_u^{1/2}\{[\phi'(0) - \phi'(v)]/\phi(v)\}dv}\Big\{2 - \\ &\epsilon[(2\phi'(0) - \phi'(u))[(1/2) - u]^3 - 2\phi(u)[(1/2) - u]^2]e^{-2\int_u^{1/2}\{[\phi'(0) - \phi'(v)]/\phi(v)\}dv}\Big\}. \end{aligned} \tag{2.18}$$

Because the coefficient of ϵ in the factor in braces is bounded, we can choose ϵ so small that this factor is positive in (0,1/2). Since $K_{\tilde{\phi}} = K_\phi$ in (1/2,1), we have shown that

$$K_{\tilde{\phi}}(u) \geq K_\phi(u) \text{ in } [0,1],$$

with strict inequality in (0,1/2). Thus the failure of the condition (2.13) leads to the conclusion that the condition $f \leq f'(0)K_\phi$ is improvable, and therefore not extremal.

We have thus established all the statements of Theorem 2.3. □

We note that while $\phi'(u)$ is required to be bounded above, it may not be bounded below. This makes it possible for an extremal function K_ϕ to be bounded away from 0 at $u = 1$.

Example 2.2. For some constant γ with

$$0 < \gamma < 1$$

choose

$$\phi(u) = (1-u)^\gamma - (1-u).$$

Then

$$\phi'(u) = 1 - \gamma(1-u)^{\gamma-1},$$

so that $\phi(0) = \phi(1) = 0$, $\phi'(0) = 1-\gamma$, and

$$K_\phi(u) = (1-\gamma)^{-2}(1-u)^{2\gamma-1}[1-(1-u)^{1-\gamma}][\gamma + (1-2\gamma)(1-u)^{1-\gamma}]. \quad (2.19)$$

Since $\gamma < 1$, both of the factors in square brackets are positive in $(0,1)$, so that ϕ satisfies the conditions of Theorem 2.3. That is, the sufficient condition $f(u) \leq \mathbf{f}'(0)K_\phi(u)$ for linear determinacy is extremal.

A simple calculation shows that

$$K''_\phi(u) = \gamma(1-\gamma)^{-1}(1-u)^{2\gamma-3}[2(1-2\gamma) - (1-3\gamma)(1-u)^{1-\gamma}].$$

Therefore, $K''_\phi(u) > 0$ in (0,1] when $0 < \gamma \leq 1/2$. Since $K_\phi(0) = 0$ and $K'_\phi(0) = 1$, it follows that

$$K_\phi(u) > u \text{ when } 0 < \gamma \leq 1/2 \text{ and } 0 < u \leq 1.$$

That is, when $\gamma \leq 1/2$, the condition $\mathbf{f}(u) \leq f'(0)K_\phi(u)$ is an improvement of the classical condition $f(u) \leq f'(0)u$.

On the other hand, we note that $\gamma > 1/2$ implies that $K_\phi(0) = 0$, so that the extremal condition $f(u) \leq f'(0)K_\phi(0)$ does not improve the classical condition. □

Remarks. 1. The particular case $\gamma = 1/2$ of (2.19) is (1.4) in the Introduction.

2. Note that $K_\phi(1) = +\infty$ when $0 < \gamma < 1/2$.

3. If the function ϕ in the above example is replaced by

$$\phi(u) = (1-u^{1+\delta})^\gamma - 1 + u$$

with $0 < \gamma < 1$ and $\delta > 0$, then K_ϕ behaves like the K_ϕ in the preceding example at $u = 1$, and like the right-hand side of (2.10) at $u = 0$. In particular, if $\delta > 0$ and $0 < \gamma < \delta/[2(\delta+1)]$, then $K_\phi(u)/u$ is increasing at an unbounded rate near $u = 0$, while $K_\phi(1) = \infty$.

Example 2.3. Choose

$$\phi(u) = u\left\{1 - \frac{1}{1+\ln(1/u)}\right\}.$$

Then

$$K_\phi(u) = u\left\{1 - \frac{1}{[1+\ln(1/u)]^3}\right\}.$$

The condition $f(u) \leq f'(0)K_\phi(u)$ is clearly improved by the classical condition $f \leq u$, even though $\phi(1) = 0$. This is consistent with Theorem 2.3 because the condition (2.13) is violated. □

Integrating the definition (2.7) and using the fact that $\phi'(u) < 2\phi'(0)$ shows that

$$\int_0^1 K_\phi(u)du \leq 1,$$

However, this averaging result does not prevent K_ϕ from being large at some points.

Example 2.4. Choose a u_0 in $(0, 1)$, and define

$$\phi(u) := u[\arctan \alpha(1 - u_0) - \arctan \alpha(u - u_0)].$$

It is easily verified that ϕ satisfies the conditions (2.6) and (2.13), and that $\phi(1) = 0$. The corresponding extremal function is

$$\begin{aligned} K_\phi(u) = \phi(u)\{&\arctan \alpha(1 - u_0) + 2 \arctan \alpha u_0 + \arctan \alpha(u - u_0) \\ &+ \alpha u/[1 + \alpha^2(u - u_0)^2]\}/\{\arctan \alpha u_0 + \arctan \alpha(1 - u_0)\}^2. \end{aligned}$$

We observe that $K_\phi(u)$ can be made arbitrarily large on the interval $|u - u_0| \leq 1/\alpha$ by choosing α sufficiently large. On the other hand, K_ϕ remains uniformly bounded on the set $|u - u_0| \geq \alpha^{-1/2}$. Thus for large α, K_ϕ has a large peak at u_0.

By choosing ϕ to be a convex linear combination of these functions with different values of u_0, we can obtain an extremal function which has arbitrarily large peaks on any prescribed finite set of points of the interval $(0, 1)$. □

3. **Conditions for the violation of linear determinacy.** Just as the upper bound of Hadeler and Rothe [3] leads to to a sufficient condition for the spreading speed of (1.1) to be linearly determinate, another theorem of Hadeler and Rothe [3] leads to a sufficient condition for the failure of (1.1) to be linearly determinate.

Theorem 3.1. *Choose any function ϕ which satisfies the conditions (2.6), and the additional conditions*

$$\phi(1) = 0, \tag{3.1}$$

$$\sup_{0 \leq u < 1} \phi'(u) < (2 - \epsilon)\phi'(0) \text{ for some constant } \epsilon \text{ with } 0 < \epsilon < 1, \tag{3.2}$$

and

$$\lim_{u \nearrow 1} \phi(u)\phi'(u) = 0. \tag{3.3}$$

Define the function

$$L_{\phi,\epsilon}(u) := \phi(u)[(2 - \epsilon)\phi'(0) - \phi'(u)]/[(1 - \epsilon)\phi'(0)^2]. \tag{3.4}$$

Then the condition

$$f(u) \geq f'(0)L_{\phi,\epsilon}(u) \text{ for } 0 \leq u \leq 1 \tag{3.5}$$

implies that the spreading speed c^* of the equation (1.1) satisfies the inequality

$$c^* \geq \sqrt{f'(0)}(2 - \epsilon)/\sqrt{1 - \epsilon}. \tag{3.6}$$

Moreover,

$$(2 - \epsilon)/[\sqrt{1 - \epsilon} > 2, \tag{3.7}$$

so that the spreading speed of (1.1) is not linearly determinate.

Proof. Define the function

$$\hat{f}(u) := f'(0)L_{\phi,\epsilon}(u), \tag{3.8}$$

and the number

$$\hat{c} := \sqrt{f'(0)}(2-\epsilon)/\sqrt{1-\epsilon}. \tag{3.9}$$

We note the identity

$$\left[(2-\epsilon)/\sqrt{1-\epsilon}\right]^2 = 4 + \left[\epsilon^2/(1-\epsilon)\right]. \tag{3.10}$$

This immediately gives the inequality (3.7), so that

$$\hat{c} > 2\sqrt{f'(0)}. \tag{3.11}$$

The inequality (3.2) and the assumptions (2.6), (3.1), and (3.3) show that $L_{\phi,\epsilon} > 0$ in (0,1) and $L_{\phi,\epsilon}(1) = 0$, so that $\hat{f}(0) = \hat{f}(1) = 0$ and $\hat{f} > 0$ in $(0,1)$. The definition (3.4) shows that

$$L'_{\phi,\epsilon}(0) = 1, \tag{3.12}$$

so that

$$\hat{f}'(0) = f'(0).$$

If we define the new function

$$\hat{\phi}(u) := [\sqrt{f'(0)}\phi(u)]/[\sqrt{1-\epsilon}\phi'(0)],$$

the definition (3.4) can be written in the form

$$\hat{\phi}(u)\hat{\phi}'(u) - \hat{c}\hat{\phi}(u) + \hat{f}(u) = 0. \tag{3.13}$$

This is the phase-plane equation of a traveling wave of speed $\hat{c}$ for the equation (1.1) with f replaced by $\hat{f}$. Moreover, we see from (3.10 that

$$\hat{\phi}'(0) = \sqrt{f'(0)}/\sqrt{1-\epsilon} = (1/2)[\hat{c} + \sqrt{\hat{c}^2 - 4f'(0)}]. \tag{3.14}$$

The plus sign on the right shows that the phase-plane trajectory $w' = -\phi(w)$ lies below all the other trajectories which approach the origin in the fourth quadrant. Theorem 7 of Hadeler and Rothe [3] states that this implies that $\hat{c}$ is the spreading speed of the equation (1.1) with f replaced by $\hat{f}$. For the sake of completeness, we sketch the proof of this statement.

We suppose for the sake of contradiction that there is a traveling wave $w(x-\tilde{c}t)$ of the equation (1.1) with f replaced by $\hat{f}$ for some $\tilde{c}$ with

$$2\sqrt{f'(0)} \leq \tilde{c} < \hat{c}.$$

Then the corresponding phase plane equation has a trajectory $w' = -\tilde{\phi}(w)$, and $\tilde{\phi}$ satisfies the equation

$$\tilde{\phi}(u)\tilde{\phi}'(u) - \tilde{c}\tilde{\phi}(u) + \hat{f}(u) = 0 \tag{3.15}$$

and the conditions

$$\tilde{\phi}(0) = \tilde{\phi}(1) = 0 \text{ and } \tilde{\phi}(u) > 0 \text{ in } (0,1).$$

By differentiating the equation and setting $u = 0$, we obtain a quadratic equation for $\tilde{\phi}'(0)$ whose solution is

$$\tilde{\phi}'(0) = (1/2)[\tilde{c} \pm \sqrt{\tilde{c}^2 - 4\hat{f}'(0)}.$$

Because $\hat{f}'(0) = f'(0)$, $\tilde{c} < \hat{c}$, and the identity (3.14) is valid, we find that

$$\tilde{\phi}'(0) < \hat{\phi}'(0).$$

Since both $\tilde{\phi}$ and $\hat{\phi}$ vanish at zero, we find that $\hat{\phi}(u)-\tilde{\phi}(u)$ is positive in some open interval whose left end point is 0.

In the interval $(0,1)$ we solve (3.13) for $\hat{\phi}'(u)$ and (3.15) for $\tilde{\phi}'(u)$ and subtract to find that

$$[\hat{\phi}(u)-\tilde{\phi}(u)]' = \hat{c}-\tilde{c}+[\hat{\phi}(u)-\tilde{\phi}(u)]\hat{f}(u)/[\hat{\phi}(u)\tilde{\phi}(u)].$$

In particular, this shows that

$$[\hat{\phi}(u)-\tilde{\phi}(u)]' \geq \hat{c}-\tilde{c} > 0 \text{ when } \hat{\phi}(u)-\tilde{\phi}(u) \geq 0 \tag{3.16}$$

This inequality says that if $\hat{\phi}(u)-\tilde{\phi}(u)$ vanishes at a point in $(0,1)$, then this function is negative immediately to the to the left of this point. Therefore, the interval with left end point 0 where $\hat{\phi}-\tilde{\phi}>0$ cannot end in $(0,1)$. Thus, $\hat{\phi}-\tilde{\phi}$ is positive in $(0,1)$. Hence the inequality (3.16) holds in $[0,1]$. Integration then shows that

$$\hat{\phi}(u) \geq \tilde{\phi}(u)+[\hat{c}-\tilde{c}]u$$

Since $\tilde{\phi}\geq 0$ and $\tilde{c}<\hat{c}$, this inequality would contradict the fact that

$$\hat{\phi}(1)=\sqrt{f'(0)}\phi(1)/[\sqrt{1-\epsilon}]\phi'(0)]=0.$$

This contradiction shows that if $\tilde{c}<\hat{c}$, the equation (1.1) with f replaced by $\hat{f}$ has no nonincreasing traveling wave solution $w(x-\tilde{c}t)$ with $w(-\infty)=1$ and $w(\infty)=0$. That is, $\hat{c}$ is the minimum of the speeds of such wave. Hence $\hat{c}$ is also the spreading speed of the equation (1.1) with $f=f'(0)L_{\phi,\epsilon}$. This is the statement of Theorem 7 of [3].

A simple comparison argument shows that increasing the function f does not decrease the spreading speed c^* of (1.1). This and the above result show that if $f\geq f'(0)L_{\phi,\epsilon}$, then $c^*\geq\hat{c}$. We have thus proved Theorem 3.1. ☐

Remarks. 1. The special case of this theorem with $\phi(u)=u(1-u)$ and $\epsilon=(\nu-2)/\nu$ was used by Hadeler and Rothe [3] to show that when $\nu>2$, the equation (1.1) with $f(u)=u(1-u)(1+2\nu)$ has the spreading speed $c^*=\sqrt{\nu/2}+\sqrt{2/\nu}$.

2. Once (3.2) is satisfied, the conditions (3.1) and (3.3) are equivalent to the property that $L_{\phi,\epsilon}(1)=0$. If $L_{\phi,\epsilon}$ does not have this property, there is no f with $f\geq f'(0)L_{\phi\epsilon}$ and $f(1)=0$, so that the statement of Theorem 3.1 becomes vacuous.

Example 3.1. Choose

$$\phi(u)=u(1+u^{1/2}-2u).$$

Then the condition (3.2) is satisfied when $\epsilon<55/64$. We choose $\epsilon=1/2$, and calculate

$$L_{\phi,1/2}=u(1+u^{1/2}-2u)(1-(3/2)u^{1/2}+4u).$$

Because the function $L_{\phi,1/2}(u)/u$ behaves like $1-(1/2)u^{1/2}$ near 0, its derivative approaches $-\infty$ at 0, so that $L_{\phi,1/2}/u$ is strictly decreasing for all sufficiently small u. Thus for the function $f(u)=L_{\phi,1/2}$ there is no Allee effect, but the spreading speed of the equation (1.1) is not linearly determinate. ☐

We note that if the condition (3.2) is satisfied for one value of ϵ, it is also satisfied for any smaller positive value of ϵ. The definitions (3.4) and (2.7) show that

$$L_{\phi,\epsilon}(u)=K_\phi(u)+[\epsilon/(1-\epsilon)]\phi(u)[\phi'(0)-\phi'(u)]/\phi'(0)^2. \tag{3.17}$$

Therefore, if the function $\phi'(0)-\phi'(u)$ takes on both positive and negative values, then the conditions $f\leq f'(0)L_{\phi,\epsilon}$ for different values of ϵ are not comparable. However, if, as in Examples 2.1 and 2.2, $\phi'(u)<\phi'(1)$ in $(0,1)$, then the inequality (3.5) becomes less stringent when ϵ is reduced to another positive value. It is then worthwhile to look at the limiting behavior of this inequality as ϵ decreases to 0. Of course, $L_{\phi,\epsilon}$ converges to K_ϕ as ϵ goes to 0.

Theorem 3.2. *Suppose that $\phi(u)$ satisfies the conditions of Theorem 3.1, and that, in addition,*

$$\phi'(u) < \phi'(0) \text{ for } 0 < u < 1. \tag{3.18}$$

Let $f(u)$ satisfy the conditions (2.2) and the three additional conditions

$$\begin{aligned} &f(u) > f'(0)K_\phi(u) \text{ for } 0 < u < 1, \\ &\liminf_{u\searrow 0} \frac{f(u) - f'(0)K_\phi(u)}{\phi(u)[\phi'(0) - \phi'(u)]} > 0, \text{ and} \\ &\liminf_{u\nearrow 1} \frac{f(u) - f'(0)K_\phi(u)}{\phi(u)[\phi'(0) - \phi'(u)]} > 0. \end{aligned} \tag{3.19}$$

Then the spreading speed c^* of (1.1) is not linearly determinate. That is, $c^* > 2\sqrt{f'(0)}$.

Proof. By (3.17) we can write the condition $f(u) \geq f'(0)L_{\phi,\epsilon}(u)$ in the form

$$f(u) - f'(0)K_\phi(u) \geq [\epsilon/(1-\epsilon]f'(0)\phi(u)[\phi'(0) - \phi'(u)]/\phi'(0)^2.$$

Because the right-hand side is positive in $(0,1)$, there is a positive ϵ for which this inequality is satisfied if and only if the ratio of the left-hand side to the function $\phi(u)[\phi'(0) - \phi'(0)]$ is uniformly positive. The three conditions (3.19) are clearly necessary and sufficient for this to be the case. □

Remarks. 1. Although the same function K_ϕ appears in both Theorems 2.1 and 3.2, it must be remembered that inequalities give only a partial ordering of functions. That is, there are many functions which satisfy neither $f \leq f'(0)K_\phi$ nor $f \geq f'(0)K_\phi$.

2. As usual, the denominator in the last two inequalities can be replaced by any function with the same asymptotic behavior at 0 or at 1. Moreover, K_ϕ can then be replaced by any approximation with the property that the ratio of its difference from K_ϕ to the denominator approaches 0.

Example 3.2. Choose

$$\phi(u) = u(1-u).$$

Then

$$K_\phi(u) = u(1 + u - 2u^2),$$

and

$$\phi'(0) - \phi'(u) = 2u.$$

Thus, $\phi(u)[\phi'(0) - \phi'(0)$ behaves like $2u^2$ near 0 and like $2(1-u)$ near 1. Therefore the three conditions in Theorem 3.2 become

$$f(u) > u(1 + u - 2u^2) \text{ in } (0,1),$$

$$\liminf_{u\searrow 0} \{[f(u) - f'(0)(u + u^2)]/u^2\} > 0,$$

and

$$\liminf_{u\nearrow 1} \{[f(u) + 3f'(0)(1-u)]/[1-u]\} > 0.$$

If f is twice differentiable at 0, the condition at 0 becomes

$$f''(0) > 2f'(0).$$

If f is differentiable at 1, the condition at 1 becomes

$$f'(1) < -3f'(0).$$

□

Because the condition $f \geq L_{\phi,\epsilon}$ becomes less stringent when $L_{\phi,\epsilon}$ is decreased, it can be useful to apply Theorem 3.2 to functions ϕ for which K_ϕ is not extremal.

Example 3.3. As in Example 2.3, we let

$$\phi(u) = u\left\{1 - \frac{1}{1 + \ln(1/u)}\right\}.$$

Then

$$K_\phi(u) = u\left\{1 - \frac{1}{[1 + \ln(1/u)]^3}\right\},$$

and

$$\phi(u)[\phi(u)\phi'(0) - \phi'(u)] = u\left\{\frac{1}{1 + \ln(1/u)} - \frac{1}{[1 + \ln(1/u)]^3}\right\}.$$

Thus, the conditions of Theorem 3.2 become

$$f(u) > u\left\{1 - \frac{1}{[1 + \ln(1/u)]^3}\right\} \text{ in } (0,1),$$

$$\liminf_{u \searrow 0}\{\ln(1/u)[f(u) - f'(u)u]/u\} > 0,$$

and

$$\liminf_{u \nearrow 1}\{[f(u) - 3f'(0)()1 - u)]/[1 - u]\} > 0.$$

If f is differentiable at 1, the condition at 1 says that $f'(1) < -3f'(0)$. However, the condition at 0 implies that $f''(0)$ does not exist. □

4. **Density-dependent diffusion and advection.** It was observed by B. H. Gilding and R. Kersner [2] that the bounds of Hadeler and Rothe are easily extended to the density-dependent reaction-advection-diffusion equation

$$u_t = d(u)u_{xx} - a(u)u_x + f(u) \tag{4.1}$$

with $d(u) > 0$, $f(0) = f(1) = 0$, and $f > 0$ in $(0,1)$. In this case, the lower bound for the rightward spreading speed c^* which is obtained by linearizing the equation is

$$c^* \geq 2\sqrt{d(0)f'(0)} + a(0).$$

The Hadeler-Rothe bound says that if for a suitable function ϕ

$$d(u)\phi(u)\phi'(u) - [c - a(u)]\phi(u) + f(u) \leq 0 \text{ in } (0,1),$$

then

$$c^* \leq c.$$

By setting $c = 2\sqrt{d(0)f'(0)} + a(0)$ and proceeding as in the proof of Theorems 2.1 and 2.3, we obtain the following extension.

Theorem 4.1. *Let the function $\phi(u)$ satisfy the conditions*

$$\phi(0) = 0,\ \phi(u) > 0 \textit{ in } (0,1),\ \textit{and}$$
$$d(u)\phi'(u) + a(u) < 2d(0)\phi'(0) + a(0) \textit{ in } (0,1).$$

Define the function

$$K_\phi(u) := \phi(u)[2d(0)\phi'(0) + a(0) - d(u)\phi'(u) - a(u)]/[d(0)\phi'(0)^2]. \tag{4.2}$$

Then if

$$f(0) = f(1) = 0,\ f(u) > 0 \textit{ in } (0,1), \textit{ and}$$
$$f(u) \le f'(0)K_\phi(u) \textit{ in } [0,1],$$

the rightward spreading speed of the reaction-advection-diffusion equation (4.1) is linearly determinate. That is, $c^* = 2\sqrt{d(0)f'(0)} + a(0)$.

Moreover, the condition $f \le f'(0)K_\phi$ *is extremal if and only if* $\phi(1) = 0$*, and*

$$\liminf_{u\searrow 0} \int_u^{1/2} \frac{2d(0)\phi'(0) + a(0) - 2d(v)\phi'(v) - a(v)}{d(v)\phi(v)} dv < +\infty.$$

The obvious extension of the proof of Theorem 3.1 shows that the statement of this theorem is still valid for the equation (4.1), provided the condition (3.2) is replaced by

$$d(u)\phi'(u) + a(u) < (2-\epsilon)d(0)\phi'(0) + a(0),$$

and the definition (3.4) of $L_{\phi,\epsilon}$ is replaced by

$$\begin{aligned} L_{\phi,\epsilon}(u) :=& \phi(u)\big\{(2-\epsilon)d(0)\phi'(0) + a(0) \\ & - d(u)\phi'(u) - a(u)]\big\}/\{(1-\epsilon)d(0)\phi'(0)^2\}. \end{aligned}$$

The statement of Theorem 3.2 remains valid when the the function $\phi'(0) - \phi'(u)$ is replaced by $d(0)\phi'(0) + a(0) - d(u)\phi'(u) - a(u)$.

Example 4.1. Consider the family of equations

$$u_t = (1+u)u_{xx} - \alpha(1-3u)u_x + f(u) = 0.$$

Choose

$$\phi(u) = u(1-u).$$

Then

$$K_\phi(u) = u(1-u)[1 + (1+3\alpha)u + 2u^2],$$
$$d(0)\phi'(0) + a(0) - d(u)\phi'(u) - a(u) = (1+3\alpha)u + 2u^2,$$

and

$$L_{\phi,\epsilon} = K_\phi(u) + [\epsilon/(1-\epsilon)]u(1-u)[(1+3\alpha)u + 2u^2].$$

K_ϕ and $L_{\phi.\epsilon}$ are positive in $(0,1)$ so that the extensions of Theorems 2.1, 2.3, and 3.1 can be applied when

$$\alpha > -[2\sqrt{2} + 1]/3.$$

The extension of Theorem 3.2 can only be applied when

$$\alpha \ge -1/3.$$

□

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