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Boundary and initial boundary-value problems for separable backward-forward parabolic problems.

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To Hugh Turrittin on his 90th birthday

1. Introduction. This work is concerned with solutions of a partial differential equation of the form $-T_{xx} + b(x)T_y = 0$ where the coefficient $b(x)$ changes sign. More specifically, we shall consider the solution, by separation of variables, of backward-forward boundary value problems of the form

$$\begin{aligned}
&-T_{xx} + b(x)T_y = 0 \quad \text{for } -1 < x < 1, -\infty < y < \infty \\
&T(1,y) = 0 \quad \text{for } -\infty < y < \infty \\
&T(-1,y) = f(y) \quad \text{for } -\infty < y < \infty \\
&T(x,y) \text{ bounded}
\end{aligned} \tag{1.1}$$

on the strip $[-1,1] \times (-\infty, \infty)$, and initial-boundary value problems of the form

$$\begin{aligned}
&-T_{xx} + b(x)T_y = 0 \qquad \text{for } -1 < x < 1,\ y > 0 \\
&T(x,0) = u(x) \qquad \text{for } x \in \Gamma^+ = \{x : b(x) > 0\} \\
&T_x(-1,y) = T_x(1,y) = 0 \qquad \text{for } y > 0 \\
&u \text{ bounded}
\end{aligned} \tag{1.2}$$

on the half-strip $[-1,1] \times [0,\infty)$.

In studying the steady advection and diffusion of heat in parallel flow in a channel, Ludford and Robertson [LR] were led to a whole channel boundary value problem of the form (1.1) for the temperature $T(x,y)$. Here $b(x)$ is the given velocity of the flow, which changes sign within the channel. Thus (1.1) is a backward-forward parabolic partial differential equation. Ludford and Robertson solved the problem for the case when $b(x)$ and $f(y)$ are step functions by using a matching technique. We shall present a very simple method of solving it for a large class of $b(x)$ and $f(y)$.

By a **bounded weak solution** of the problem (1.1) we mean a bounded measurable function $T(x,y)$ such that the integration by parts formula

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-1}^{1} T(x,y)[-h_{xx} - b(x)h_y]dxdy \\
&\qquad = \int_{-\infty}^{\infty} f(y)h_x(-1,y)dy
\end{aligned} \tag{1.3}$$

is valid for all smooth functions h which vanish on the boundary $x = \pm 1$, and which vanish for all y with sufficiently large absolute value.

Because the function $b(x)$ is independent of y, the problem (1.1) is separable. That is, the differential equation and the boundary conditions are satisfied by the product functions $\psi_n(x)e^{-\lambda_n y}$, where the λ_n and ψ_n are the eigenvalues and eigenfunctions of the problem

$$\begin{aligned} -\psi'' &= \lambda b \psi \\ \psi(-1) &= \psi(1) = 0. \end{aligned} \tag{1.4}$$

Because b changes sign, this problem has infinitely many positive eigenvalues, which we write in increasing order as λ_n with $n > 0$, and infinitely many negative eigenvalues, which we write in increasing order as λ_n with $n < 0$.

We shall obtain the following result, in which the hypotheses on f are tailored to include the step function treated by Ludford and Robertson [LR]. Throughout this work we shall use the notation (p, q) for the standard L_2 scalar product $\int_{-1}^{1} pq dx$.

Theorem 1. *Let b be bounded and measurable, and let its null set have measure zero. Let f be bounded, and suppose that there are at most finitely many points at which f has jump discontinuities, and that in the intervals outside these jumps f is uniformly Lipschitz continuous. Then the problem (1.1) has a unique bounded weak solution, which is given by the Fourier series*

$$\begin{aligned} T(x,y) =& \frac{1}{2} f(y)(1-x) - \sum_{n<0} \frac{\psi_n'(-1)}{(b\psi_n, \psi_n)} \left\{ \int_y^\infty [f(z) - f(y)] e^{\lambda_n (z-y)} dz \right\} \psi_n(x) \\ &+ \sum_{n>0} \frac{\psi_n'(-1)}{(b\psi_n, \psi_n)} \left\{ \int_{-\infty}^y [f(z) - f(y)] e^{\lambda_n (z-y)} dz \right\} \psi_n(x). \end{aligned} \tag{1.5}$$

This series converges uniformly on every closed set which does not contain a point (x, y) with y a jump point of f. The solution satisfies the inequalities

$$\min\{\inf[f], 0\} \leq T(x,y) \leq \max\{\sup[f], 0\}.$$

If y_0 is a point of continuity of f, then $T(x, y)$ is continuous in (x, y) on the line $y = y_0$, and the function $T(\cdot, y_0)$ is in $H^1[-1, 1]$ and assumes the boundary values $f(y_0)$ at -1 and 0 at 1.

If y_0 is a jump point of f, then T is continuous in (x, y) in any open subinterval of the line $y = y_0$ in which either $b \geq \alpha$ or $b \leq -\alpha$ for some positive α.

Initial-boundary problems of the form (1.2) with various boundary conditions arise in a variety of applications, such as diffusion governed by the steady-state Fokker-Planck equation and electron scattering. See [GvdMP] for an overview of applications and results.

By a **bounded weak solution** T of the problem (1.2) we mean a bounded measurable function which satisfies the equation

$$\int_0^\infty \int_{-1}^1 T(x,y)[-h_{xx} - b(x) h_y] dx dy = \int_{\Gamma^+} u(x) h(x, 0) dx \tag{1.6}$$

for all smooth functions h which satisfy the boundary conditions $h_x(\pm 1, y) = 0$, which vanish for all sufficiently large y, and which vanish on the part of the initial line $y = 0$ where $b < 0$.

It was shown by Beals [B2, B4] that if b is continuous, if at each zero b behaves like $|x - x_0|^\alpha (x - x_0)$ for some $\alpha > -1/2$, and if the initial function u lies in the weighted L_2 space $L_2(\Gamma^+, bdx)$ then there exists a unique weak solution $T \in C([0,\infty), L_2([-1,1], |b|dx)$ $\cap L_{2,\mathrm{loc}}(H^1([-1,1])$. (For the special case of the Dirichlet problem with a linear function b, this theorem was proved earlier by Baouendi and Grisvard [BG].)

The problem (1.2) is also separable. That is, the differential equation and the boundary conditions are satisfied by products of the form $\phi_n(x)e^{-\lambda_n y}$, where the λ_n are the eigenvalues and ϕ_n the corresponding eigenfunctions of the problem

$$\begin{aligned} &-\phi'' = \lambda b(x)\phi \quad \text{for } -1 < x < 1 \\ &\phi'(-1) = \phi'(1) = 0. \end{aligned} \tag{1.7}$$

Because b changes sign, this problem again has both positive eigenvalues called λ_n with $n > 0$ and negative eigenvalues λ_n with $n < 0$. Because of the Neumann boundary conditions, it has the additional eigenvalue $\lambda_0 = 0$ with the eigenfunction $\phi_0 = 1$. If the integral of b is zero, there is also a generalized eigenfunction q which corresponds to the eigenvalue 0. (See [B3].)

Beals showed that the eigenfunctions, and the generalized eigenfunction q, if it exists, form a basis for the weighted space $L_2([-1,1], |b|dx)$. It is then easily seen (see Lemma 3.1) that for each fixed $y \geq 0$ such a weak solution can be expanded in a Fourier series in which only the coefficients of the eigenfunctions corresponding to the nonnegative eigenvalues differ from zero. In particular, one can expand the initial function u on the set Γ^+ in such an expansion, which is then a partial-range expansion.

Because the eigenfunctions do not satisfy any orthogonality relation on the set Γ^+, one cannot find explicit expressions for the coefficients c_n of this expansion in terms of u without solving the problem first. However, Beals has shown [B3] that the eigenfunctions ϕ_n with $n \geq 0$ have the property that

$$\sum_{n\geq 0} c_n^2 |(b\phi_n, \phi_n)| \leq \int_{\Gamma^+} \{\sum_{n\geq 0} c_n\phi_n(x)\}^2 |b(x)|dx. \tag{1.8}$$

It follows that when least squares approximation is used to obtain a finite sum $\sum_{n=0}^N d_n\phi_n$ which approximates u in the $L_2([-1,1], |b|dx)$ norm, this norm can be made arbitrarily small by choosing N sufficiently large, and the resulting d_n, defined to be 0 for $n > N$, are close to the correct c_n in the sense that $\sum(d_n - c_n)^2|(b\phi_n, \phi_n)|$ is small. This fact, together with an argument we shall use in the proof of Lemma 3.1, shows that the sum $\sum_{n=0}^N d_n\phi_n(x)e^{-\lambda_n y}$ gives a uniform approximation to T on any half-strip $[-1,1] \times [\alpha, \infty)$ with $\alpha > 0$. However, the error bound grows to infinity as α approaches zero, so that this method does not give good information about the solution near $y = 0$.

We shall show that when the initial function is continuous, this phenomenon can be avoided by obtaining an approximation result in the maximum norm. The solutions of

(1.2) take their maxima on $\overline{\Gamma^+} \times \{0\}$ where $\overline{\Gamma^+}$ is the closure of Γ^+. Therefore a uniform approximation of the initial function u by the sum $\sum_{n=0}^N d_n \phi_n(x)$ leads to the uniform approximation of the solution T by the sum $\sum_{n=0}^N d_n \phi_n(x) e^{-\lambda_n y}$. Of course, this uniform approximation is only possible if u is continuous.

Theorem 2. *Suppose that b is infinitely differentiable on [-1,1]. Also assume that the zeros of b are all interior points and have finite order, and that none of these zeros is a local maximum of b. If $u(x)$ is continuous on $\overline{\Gamma^+}$, then for any $\epsilon > 0$ there exists a finite set of coefficients $\{d_1, \cdots, d_N\}$ with the property that*

$$|u(x) - \sum_{n=0}^{N} d_n \phi_n(x)| \leq \epsilon \quad \text{for } x \in \Gamma^+. \tag{1.9}$$

Moreover, the solution of (1.1) satisfies the inequality

$$|T(x,y) - \sum_{n=0}^{N} d_n \phi_n(x) e^{-\lambda_n y}| \leq \epsilon. \tag{1.10}$$

Theorem 2 will be based on the following Lemma, which is of interest in itself. We shall say that $T(x,y)$ is a **bounded local weak solution** of the differential equation and boundary conditions in (1.2) in the strip $[-1,1] \times (0,\infty)$ if it is bounded and measurable and if the equation (1.6) holds for all smooth functions h which satisfy the Neumann boundary conditions and which vanish at $y = 0$ and outside a bounded set. We shall show that such a function is a classical solution of the differential equation for $y > 0$.

Lemma 3. *Let $b(x)$ be bounded and measurable, and suppose that the set where it vanishes has measure 0. Let $T(x,y)$ be a bounded local weak solution in $[-1,1] \times (0,\infty)$ of the boundary value problem (1.2). Then for all $y > 0$ the functions T, T_x, and T_{xx}/b are continuous and infinitely differentiable with respect to y, and the partial differential equation and boundary conditions in (1.2) are satisfied. Moreover, there exist coefficients c_n such that the series*

$$\sum_{n=0}^{N} c_n \phi_n e^{-\lambda_n y} \tag{1.11}$$

converges to T, uniformly on any strip $[-1,1] \times [\alpha, \infty)$ with $\alpha > 0$.

Theorem 1 will be proved in Section 2. Sections 3 and 4 are devoted to the proof of Lemma 3 and Theorem 2, respectively. Section 5 discusses what happens when other boundary condition are used in the problems (1.2) and (1.1)

2. Heat advection and diffusion in a channel. We apply the definition (1.3) of weak solution with $h = \zeta(y)\psi_n(x)e^{\lambda_n y}$, where ψ_n is an eigenfunction of the problem (1.4), and ζ vanishes outside a bounded interval. We obtain the formula

$$\int_{-\infty}^{\infty} \left\{ \zeta'(y) e^{\lambda_n y} (bT(\cdot, y), \psi_n) + \zeta(y)\psi_n'(-1) f(y) e^{\lambda_n y} \right\} dy = 0.$$

for all functions ζ which vanish outside a bounded interval.

This means that the coefficient of ζ is the distribution derivative of the coefficient of ζ'. In particular, the coefficient of ζ' differs from an indefinite integral of the coefficient of ζ by a constant. Because f and T are bounded, we can evaluate the constant by letting y approach infinity if $\lambda_n < 0$ and minus infinity if $\lambda_n > 0$. Thus, we obtain the explicit formulas

$$(bT(\cdot, y), \psi_n) = \begin{cases} -\psi_n'(-1) \int_y^\infty f(z) e^{\lambda_n (z-y)} dz & \text{for } n < 0 \\ \psi_n'(-1) \int_{-\infty}^y f(z) e^{\lambda_n (z-y)} dz & \text{for } n > 0. \end{cases} \tag{2.1}$$

We define the function

$$\mu(x) = \frac{1}{2}(1 - x). \tag{2.2}$$

The definition (1.4) of ψ_n and integration by parts show that

$$(b\mu, \psi_n) = (\mu, -\lambda_n^{-1} \psi_n'') = \lambda_n^{-1} \psi_n'(-1). \tag{2.3}$$

Subtracting $f(y)$ times this formula from (2.1) yields the formal Fourier series

$$\begin{aligned} &T(\cdot, y) - f(y)\mu(x) = \sum d_n(y) \psi_n(x), \\ &d_n(y) = \begin{cases} -(b\psi_n, \psi_n)^{-1} \psi_n'(-1) \int_y^\infty [f(z) - f(y)] e^{\lambda_n (z-y)} dz & \text{for } n < 0 \\ (b\psi_n, \psi_n)^{-1} \psi_n'(-1) \int_{-\infty}^y [f(z) - f(y)] e^{\lambda_n (z-y)} dz & \text{for } n > 0. \end{cases} \end{aligned} \tag{2.4}$$

Thus if the weak solution T exists, its Fourier series must have the form (1.5).

We shall consider the convergence of this series in the Hilbert space H_0^1 with the scalar product

$$(u, v)_{H_0^1} = (u', v'). \tag{2.5}$$

It was observed by Manes and Michelletti [MM] that the eigenfunctions ψ_n of (1.4) are also the eigenfunctions of the operator M, which is defined by the equation $(M[v], w)_{H_0^1} = (bv, w)$ for all $w \in H_0^1$. It is easily seen that $M[v]$ is the solution of the problem

$$\begin{aligned} -(M[v])'' &= bv \\ M[v](-1) = M[v](1) &= 0. \end{aligned} \tag{2.6}$$

Because b is bounded, this operator is compact. Because the null set of b has measure zero, M has no null space, and its eigenvalues are the reciprocals of the λ_n. In particular, the ψ_n form a complete orthogonal set in H_0^1.

Even though the function μ is not in H_0^1 because it does not vanish at -1, the problem (2.6) with $v = \mu$ still defines a function $M[\mu]$ in H_0^1. Integration by parts shows that $M[\mu]$ has the Fourier series

$$M[\mu] = \sum \frac{\psi_n'(-1)}{\lambda_n^2 (b\psi_n, \psi_n)} \psi_n. \tag{2.7}$$

This series converges to $M[\mu]$ in H_0^1.

We shall suppose for the time being that f has no jump points, so that it is uniformly Lipschitz continuous. That is, there is a constant A such that $|f(z) - f(y)| \leq A|z - y|$ for all y and z. Then a simple integration shows that

$$|d_n(y)| \leq A|\psi'(-1)||(b\psi_n, \psi_n)^{-1}|\lambda_n^{-2}. \tag{2.8}$$

This inequality shows that $|d_n|$ is bounded by A times the absolute values of the n-th Fourier coefficient of $M[\mu]$ in (2.7). Because the Fourier series of $M[\mu]$ converges in H_0^1 in which the ψ_n are orthogonal, it follows that the series in (2.4) also converges in H_0^1, uniformly in y. Thus the function T defined by (2.4) is continuous and satisfies the boundary conditions, and T is in H^1 for each y.

In order to verify that this function T is a weak solution, we consider a smooth function $h(x, y)$ which vanishes at $x = \pm 1$ and outside a bounded set. Because the series in (2.4) converges uniformly, we can write the left-hand side of (1.3) in the form

$$\begin{aligned} &\int_{-\infty}^{\infty} (-h_{xx} - bh_y, T)\, dy \\ &= \int_{-\infty}^{\infty} (-h_{xx}, f(y)\mu)\, dy - \int_{-\infty}^{\infty} f(y)\,(bh_y, \mu)\, dy + \sum \int_{-\infty}^{\infty} (-h_{xx} - bh_y, d_n\psi_n)\, dy. \end{aligned} \tag{2.9}$$

Integration by parts with respect to x and use of the definition (2.2) of μ show that $(-h_{xx}, f(y)\mu) = f(y)h_y(-1, y)$. Therefore, the first integral on the right side of (2.9) is equal to the integral on the right side of (1.3), so (1.3) will hold if the last two terms in (2.9) cancel.

Because h_y is smooth and vanishes at $x = \pm 1$, its Fourier series converges uniformly in x and y. Since $(b\mu, \psi_n)$ is given by (2.3), we find that

$$-\int_{-\infty}^{\infty} f(y)(bh_y, \mu)dy = -\sum \lambda_n^{-1}\psi_n'(-1)(b\psi_n, \psi_n)^{-1} \int_{-\infty}^{\infty} f(y)(bh_y, \psi_n)dy\ . \tag{2.10}$$

Next we use integration by parts and the definition (1.4) of ψ_n to show that

$$(-h_{xx} - bh_y, \psi_n) = \left[\lambda_n - \frac{d}{dy}\right](bh, \psi_n)\ .$$

By using this relation and the definition (2.4) of d_n, we can write a term of the sum in

(2.9) with $n < 0$ as follows:

$$
\begin{aligned}
&-(b\psi_n,\psi_n)^{-1}\psi_n'(-1)\int\limits_{-\infty}^{\infty}\int\limits_{y}^{\infty}[f(z)-f(y)]\,e^{\lambda_n(z-y)}dz\left[\lambda_n-\frac{\partial}{\partial y}\right](bh(\cdot,y),\psi_n)dy\\
&=-(b\psi_n,\psi_n)^{-1}\psi_n'(-1)\left\{\int\limits_{-\infty}^{\infty} f(z)\int\limits_{-\infty}^{z}\left(-\frac{\partial}{\partial y}\right)e^{\lambda_n(z-y)}(bh(\cdot,y),\psi_n)dydz\right.\\
&\qquad\left.+\lambda_n^{-1}\int\limits_{-\infty}^{\infty} f(y)\left[\lambda_n(bh,\psi_n)-(bh_y,\psi_n)\right]dy\right\}\\
&=\lambda_n^{-1}(b\psi_n,\psi_n)^{-1}\psi_n'(-1)\int\limits_{-\infty}^{\infty} f(y)(bh_y,\psi_n)dy\ .
\end{aligned}
\tag{2.11}
$$

The same expression holds for $n > 0$. We see that these terms are exactly the negatives of those in (2.10), so that the last two terms in (2.9) cancel. Thus T given by (2.4) is indeed a weak solution of the problem (1.1) under the additional hypothesis that f is continuous.

In order to obtain bounds for T, we first make the further temporary assumption that there exists a constant B such that $f(y) = 0$ for $|y| \geq B$. Let F be a bound for $|f|$. Then if $y > B$, we see from the formulas in (2.4) that $d_n(y) = 0$ for $n < 0$, and that for $n > 0$

$$|d_n(y)| \leq F|\psi_n'(-1)||\lambda_n^{-2}|(b\psi_n,\psi_n)^{-1}|e^{\lambda_n(B-y)}.$$

The last factor approaches zero exponentially as $y \to \infty$, uniformly in n. Thus comparison with the series (2.7) for $M[\mu]$ shows that the convergent series in (2.4) approaches zero as $y \to \infty$, uniformly in x. The same argument gives the same result as $y \to -\infty$.

Thus, if f is uniformly Lipschitz continuous and vanishes near infinity, the corresponding weak solution of the problem (1.1) is given by (2.4), and it vanishes at infinity. The usual maximum principle then gives the bounds

$$\min\{\inf[f],0\} \leq T \leq \max\{\sup[f],0\}. \tag{2.12}$$

In other words, the statements of Theorem 1 are valid for such a function.

We shall show in the Appendix that if f satisfies only the hypotheses of Theorem 1, one can construct a sequence of functions $f_\nu(y)$ which also satisfy the temporary conditions, and which have the property that the corresponding sequence T_ν of bounded weak solutions converges, uniformly on closed rectangles which do not contain jump points, to the function T defined by (2.4). We shall also show that this function has the properties stated in Theorem 1.

3. A regularity lemma. In addition to the positive eigenvalues λ_n with $n > 0$ and the negative eigenvalues λ_n with $n < 0$, the problem (1.7) has the eigenvalue $\lambda_0 = 0$ with the eigenfunction 1. If $(b1, 1) = 0$, there is also a generalized eigenfunction q, which is a solution of the problem

$$-q'' = b \quad \text{for } -1 < x < 1, \quad q'(-1) = q'(1) = 0 \tag{3.2}$$

with the normalization

$$(bq, q) = 0. \tag{3.3}$$

(Note that $(b1, q) = (q', q') > 0$. Therefore there is a unique constant which must be added to a solution of the differential equation in (3.2) to make the sum satisfy (3.3).) The function q is a generalized eigenfunction of (1.7) corresponding to $\lambda_0 = 0$.

It was pointed out by Beals [B4, p. 403] that one can treat this problem like the Dirichlet problem by introducing the Hilbert space $\hat{H}$ with the scalar product

$$(v, w)_{\hat{H}} = (v', w') + \begin{cases} |(b1,1)|^{-1}(bv,1)(bw,1) & \text{if } (b1,1) \neq 0 \\ (bq,1)^{-1}(bv,q)(bw,q) & \text{if } (b1,1) = 0, \end{cases}$$

whose norm is equivalent to the H^1 norm.

One defines the compact self-adjoint operator $\hat{M}$ on this space by the equation $(\hat{M}[v], w)_{\hat{H}} = (bv, w)$ for all $w \in \hat{H}$. Then $\hat{M}[v]$ is the solution of the problem

$$-(\hat{M}[v])'' = \begin{cases} b\{v - (b1,1)^{-1}(bv,1)1\} & \text{if } (b1,1) \neq 0 \\ b\{v - (bq,1)^{-1}(bv,1)q]\} & \text{if } (b1,1) = 0 \end{cases}$$

$$(\hat{M}[v])'(-1) = (\hat{M}[v])'(1) = 0$$

$$(b\hat{M}[v], 1) = \operatorname{sgn}[(b1,1)]\ (bv,1) \quad \text{if } (b1,1) \neq 0$$

$$((b\hat{M}[v], q) = (bv,1) \quad \text{if } (b1,1) = 0.$$

It is easily seen that $\hat{M}$ has the eigenvalues λ_n with $n \neq 0$ with the corresponding eigenfunctions φ_n. If $(b1,1) \neq 0$, $\hat{M}$ has the one additional eigenvalue $\operatorname{sgn}[(b1,1)]$ with the corresponding eigenvector 1. If $(b1,1) = 0$, $\hat{M}$ has the two additional eigenvalues ± 1 with the corresponding eigenvectors $q \pm 1$.

As usual, the eigenvectors are orthogonal with respect to the scalar product of $\hat{H}$, and they are complete in this space. It follows that the eigenvectors are also orthogonal with respect to the indefinite scalar product (bv, w). Since $\hat{H}$ is a dense subspace of L_2, the eigenfunctions are also complete in L_2.

Proof of the Lemma. By hypothesis, $T(x, y)$ is a bounded local weak solution of the problem (1.2) in the sense that the equation (1.6) holds for all smooth h which satisfy the boundary conditions $h_x(\pm 1, y) = 0$, which vanish for sufficiently large y, and which vanish at $y = 0$. In particular, this equation holds when $h = \zeta(y)\phi_n(x)e^{\lambda_n y}$ where ζ is a smooth function which vanishes at $y = 0$ and for large y. Because this h satisfies the equation $-h_{xx} - bh_y = -b(x)\zeta'(y)\phi_n(x)e^{\lambda_n y}$, we find that

$$\int_0^\infty \zeta'(y)e^{\lambda_n y}(bT(\cdot, y), \phi_n)dy = 0$$

for all smooth ζ which vanish at 0 and ∞. As is well known, this implies that the coefficient of ζ' is a constant, which we call $c_n(b\phi_n, \phi_n)$ when $n \neq 0$. Hence, we obtain the formula

$$(bT(\cdot, y), \phi_n)/(b\phi_n, \phi_n) = e^{-\lambda_n y}c_n \qquad \text{for } y > 0,\ n \neq 0. \tag{3.6}$$

Since T is bounded, we see by letting $y \to \infty$ in this equation, that $c_n = 0$ when $n < 0$. If $(b1, 1) \neq 0$, the above argument also shows that the Fourier coefficient of ϕ_0 is a constant c_0, so that the Fourier series of T has the form in (1.11).

If $(b1, 1) = 0$, the above argument still shows that $(bT(\cdot, y), \phi_n)$ is zero for $n < 0$ and constant for $n = 0$. Since the function $h = \zeta(y)[y + q(x)]$, where q is the solution of (3.2), (3.3), satisfies the equation $-h_{xx} - bh_y = -b\zeta'[y+q]$, the definition (1.6) shows that $(bT(\cdot, y), y + q)$ is also constant. Since T is bounded and $(bT(\cdot, y), 1)$ is constant, we see by letting y approach infinity that the constant $(bT(\cdot, y), 1)$ is zero, and that $(bT(\cdot, y), q)$ is constant. Since $(bq, q) = 0$ and $(bq, 1) \neq 0$, the part of the Fourier series of T which involves the terms 1 and q is $(bq, 1)^{-1}[(bT, q)1 + (bT, 1)q]$. Thus the coefficient of q in the Fourier series (3.4) of $T(x, y)$ is zero and that of $\phi_0 = 1$ is a constant c_0. Thus, whether $(b, 1)$ is zero or not, the Fourier series for $T(\cdot, y)$, has the form in (1.11).

Choose any $\alpha > 0$. Because $T(\cdot, \alpha/2) \in L_2$, $\hat{M}[T(\cdot, \alpha/2)] \in \hat{H}$. That is, the series $\sum_{n \geq 0} \lambda_n^{-1} c_n e^{-\lambda_n \alpha/2} \phi_n$ converges in $\hat{H}$. Because $\lambda_n y e^{-\lambda_n(y-\alpha/2)}$ is uniformly bounded, it follows that the Fourier series (1.11) for $T(\cdot, y)$ converges in $\hat{H}$, uniformly in y for $y \geq \alpha$. We conclude that for $y > 0$ the function $T(\cdot, y)$ is in H^1 and the series also converges to it uniformly.

In fact, $(\lambda_n y)^p e^{-\lambda_n(y-\alpha/2)}$ is bounded uniformly in n and y for $y \geq \alpha > 0$ when p is any positive integer. Therefore the same argument can be applied to any partial derivative with respect to y of the Fourier series. The same is true of the series obtained by differentiating the series for T twice with respect to x and then dividing by b. This establishes the continuity properties in Lemma 3.1. The fact that T is a classical solution follows by comparing the series for T_{xx}/b and T_y, and the Lemma is proved.

4. Proof of Theorem 2. We consider the initial-boundary value problem (1.2). A result of Freidlin and Weinberger [FW] implies that if b is Lipschitz continuous in [-1,1] with $(b1, 1) \geq 0$ and measure$(\Gamma^+) > 0$, and if u is continuous on $\overline{\Gamma^+}$, then there exists a unique bounded generalized solution T of this problem, and this solution is continuous on the set $\{(x, y) : y > 0\} \cup \{(x, y) : y \geq 0,\ b(x) > 0\}$. One can see by looking at the proof that the solution is also continuous on the set $\{(x, y) : y \geq 0,\ b(x) < 0\}$. Since the set where $b = 0$ has measure zero, we see that, in particular, $bT(x, 0) \in L_1$.

If $(b1, 1) < 0$, the construction in [FW] leads to a unique bounded generalized solution with the same properties for each prescribed value of the limit at $y = \infty$.

To prove Theorem 2 we use a result of Stroock and Taniguchi [ST, Corollary 4], which states that under the hypotheses of Theorem 2, those boundary points $(z, 0)$ where $b(z) = 0$ are regular. That is, $u(z)$ is the limit of $T(x, y)$ as (x, y) approaches $(z, 0)$ through points with $y > 0$. As we have already seen, the probabilistic solution of (1.2) which was constructed in [FW] is continuous at all points with $y = 0$ where $b \neq 0$. We conclude that T is continuous on the closed strip $[-1, 1] \times [0, \infty)$. In particular, $T(\cdot, y)$ converges uniformly to $T(\cdot, 0)$ as y goes to zero.

It is known [K, Theorem 4.1] that a generalized solution as constructed in [FW] is a weak solution in the sense of the definition (1.6). By Lemma 3 the series (1.11) converges to $T(\cdot, y)$ in $\hat{H}$, uniformly in $y \geq \alpha > 0$.

The uniform convergence of $T(\cdot, y)$ to $T(\cdot, 0)$ implies that for any $\epsilon > 0$ there is a $y_\epsilon > 0$ such that $|T(x, y_\epsilon) - T(x, 0)| < \epsilon/2$. The uniform convergence of the Fourier series to $T(\cdot, y_\epsilon)$ implies that there is an N such that

$$|T(x, y_\epsilon) - \sum_{n=0}^{N} c_n \phi_n(x) e^{-\lambda_n y_\epsilon}| < \epsilon/2.$$

Thus we obtain the inequality (1.9) of Theorem 2 with $d_n = c_n e^{-\lambda_n y_\epsilon}$. The inequality (1.10) then follows from the maximum principle.

When $(b1, 1) < 0$, the construction in [FW] leads to a unique bounded continuous solution T of (1.2) for any specified value of the limit c_0 at infinity. Hence $d_0 = c_0$ may be arbitrarily specified. That is, when $(b1, 1) < 0$, the Neumann eigenfunctions ϕ_n with $n > 0$ are complete on $\overline{\Gamma^+}$ in the maximum norm. In particular, the eigenfunction 1 can be uniformly approximated on $\overline{\Gamma^+}$ by a linear combination of these eigenfunctions.

5. Other boundary conditions. We have presented the results for the initial-boundary value problem (1.2) with Neumann boundary conditions, but they are easily extended to other boundary conditions.

For example, if the boundary conditions are replaced by the Dirichlet conditions $T(-1, y) = T(1, y) = 0$, we only need to replace the space $\hat{H}$ by H_0^1 and the eigenfunctions of $\hat{M}$ by those of M in the proofs of Lemma 3 and Theorem 2 to obtain the analogous results. However, one must require the initial function u to vanish at those points of $\overline{\Gamma^+}$ which lie on the boundary $\{-1, 1\}$. One also needs to extend the results of [FW] to the Dirichlet problem, but this is easily done.

For other homogeneous boundary conditions, one only needs to use the appropriate Hilbert space H and define the compact operator m by the equation $(m[v], w)_H = (bv, w)$.

In a similar manner, one can obtain analogs of Theorem 1 for boundary value problems of the form (1.1) with other boundary conditions. The most difficult case is that of the Neumann conditions $T_x(-1, y) = f(y)$, $T_x(1, y) = 0$. The space H_0^1 and the operator M are replaced by $\hat{H}$ and $\hat{M}$, respectively. The function μ is replaced by the function $\hat{\mu}$ which satisfies the initial conditions $\hat{\mu}(1) = \hat{\mu}'(1) = 0$ and the differential equation $-\hat{\mu}'' = (b1, 1)^{-1} b$ if $(b1, 1) \neq 0$ or $-\hat{\mu}'' = (bq, 1)^{-1} bq$ if $(b1, 1) = 0$.

If $(b1, 1) \neq 0$, we find by inserting $h = \zeta(y)$ into the analog of (1.3), that $(bT(\cdot, y), 1)$ is an antiderivative of $-f(y)$. Thus one needs the additional hypothesis that $\int_0^y f(z)dz$ is uniformly bounded, and the solution is determined only to within an additive constant. In addition, $\psi_n'(-1)$ must be replaced by $-\phi_n(-1)$ in the other coefficients in the series(1.5).

When $(b1, 1) = 0$, we insert the functions $\zeta(y)[q(x) + y \pm 1]$ into the analog of (1.3) to see that $(bT, q) + (y \pm 1)(bT, 1)$ is an indefinite integral of $-[q(-1) + y \pm 1]f(y)$. It follows that $(bT, 1)$ is again an antiderivative of f, but that also $(bT(\cdot, y), q)$ is equal to $q(-1)(bT, 1)$ minus an indefinite integral of $(bT(\cdot, y), 1)$. Thus we need the hypotheses that both $\int_0^y f(z)dz$ and $\int_0^y (y - z)f(z)dz$ are uniformly bounded. The solution is again determined to within an additive constant.

The regularity conditions are proved as in Section 2. Because there are no Dirichlet conditions, and $\mu \in H^1$, its Fourier series converges in H^1. Therefore, the solution is continuous everywhere in the strip, and lies in H^1 for each y, even when f has jumps.

Similar results can be obtained for other boundary conditions as well.

Appendix: Remainder of the proof of Theorem 1. As promised at the end of Section 2, we shall show here how to extend the proof of Theorem 1 from the case in which $f(y)$ is continuous and vanishes outside a bounded set to the general case. We shall do this in two steps.

Suppose first that f is bounded and uniformly Lipschitz continuous, but that it does not vanish outside a bounded set. Define the sequence of uniformly Lipschitz continuous approximating functions f_ν by setting

$$f_\nu(y) = \begin{cases} f(y) & \text{for } |y| \leq \nu \\ 0 & \text{for } |y| \geq \nu+1 \\ \text{linear} & \text{for } \nu \leq |y| \leq \nu+1. \end{cases}$$

Since f_ν vanishes near infinity, the function T_ν obtained by replacing f by f_ν in (2.4) is a weak solution of (1.1) with f replaced by f_ν. Because $\min\{\inf[f_\nu], 0\} \geq \min\{\inf[f], 0\}$ and $\max\{\sup[f_\nu], 0\} \leq \max\{\sup[f], 0\}$, each T_ν has the bounds (2.12).

Let $d_n^{(\nu)}(y)$ denote the coefficient in (2.4) with f replaced by f_ν. Choose any y and let $\nu > |y|$. Then $f_\nu(y) = f(y)$ and $f_\nu(z) - f(z) = 0$ for $|z| \leq \nu$. It easily follows from the formula (2.4) that if F is a bound for $|f|$,

$$|d_n^{(\nu)}(y) - d_n(y)| \leq 2F|(b\psi_n, \psi_n)^{-1}\psi_n'(-1)\lambda_n^{-1}|e^{-|\lambda_n|(\nu-|y|)}.$$

Because the function ze^{-z} is bounded for $z \geq 0$, we can bound the right-hand side by a constant times $e^{-|\lambda_n|(\nu-|y|)/2}$ times the corresponding Fourier coefficient of $M[\mu]$ in (2.7). Because as $\nu \to \infty$ the exponential function approaches 0 uniformly for y in any bounded interval, we conclude that T_ν converges to T in H_0^1 uniformly in y on any bounded set. In particular, T_ν converges to T uniformly on any bounded set.

Letting ν approach infinity in the defining relation (1.3) with f replaced by f_ν and T by T_ν then shows that T is, indeed, a bounded weak solution. The bounds (2.12) follow from letting $\nu \to \infty$ in the inequality (2.12) with T replaced by T_ν. Thus Theorem 1 is established for continuous f.

If f has finitely many jump discontinuities but has the properties in Theorem 1, we choose any integer ν larger than twice the distance between any two jump points and define the approximating sequence $f_\nu(y)$ by stating that f_ν is continuous, $f_\nu(y) = f(y)$ when the distance from y to any jump point is at least ν^{-1}, and f_ν is linear on each interval of length $2/\nu$ centered at a jump point. Let $d_n^{(\nu)}(y)$ be the coefficient defined by (2.4) with f replaced by f_ν.

Because f_ν is bounded and uniformly Lipschitz continuous, the function T_ν defined by (2.4) with f replaced by f_ν is a weak solution for these boundary values, and it satisfies the bounds (2.12). Choose a point y whose distance from any jump point is at least $\delta > 0$. Then if $\nu\delta > 1$, $f_\nu(y) = f(y)$. Moreover, if F is a bound for $|f|$, then $|f_\nu(z) - f(z)| \leq 2F$, and $f_\nu(z) - f(z)$ vanishes outside intervals of length $2/\nu$ centered at the jumps points. Thus if there are J jump points, we obtain the bound

$$|d_n^{(\nu)}(y) - d_n(y)| \leq 2FJ\nu^{-1}|(b\psi_n, \psi_n)^{-1}\psi_n'(-1)|e^{-|\lambda_n|(\delta-\nu^{-1})}.$$

By using the comparison series (2.7), one finds as before that T_ν converges to T in H_0^1, uniformly on any closed set which contains no point whose y coordinate is a jump point of f.

Because T_ν satisfies (2.12), this is sufficient to permit one to take limits in the equation (1.3) with f replaced by f_ν and T replaced by T_ν. Thus, T is a weak solution and satisfies the bounds (2.12). It is defined by means of the series (2.4) for all y which are not jump points of f. The set of lines $y = y_j$ where the y_j are the jump points has measure zero in the strip, so that the values of T on this set are not needed to verify (1.3).

If either $b \geq \alpha > 0$ or $b \leq -\alpha < 0$ in an open interval (a, b) and if y_0 is a jump point of F, then T is a bounded weak solution of a uniformly parabolic equation in a neighborhood of $(a, b) \times \{y_0\}$. It follows from standard theory that the values of T on this interval can then be chosen so that T is continuous in its neighborhood. That is, the limits of T from above and from below are equal, and they define a continuous function. We define T in this way, and this establishes the last part of Theorem 1.

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