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# Some Isoperimetric Inequalities for Membrane Frequencies and Torsional Rigidity*

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## I. INTRODUCTION

Let $\Lambda$ denote the fundamental frequency of a two-dimensional membrane $G$ fixed on its boundary. Let $A$ be the area of $G$, and $L$ its perimeter. Makai [5, 6] has recently shown that if $G$ is simply or doubly connected, the dimensionless quantity $\Lambda^2 A^2 L^{-2}$ is at most 3. Pólya [7] has improved this result to

$$\Lambda^2 \leqslant (\tfrac{1}{2}\pi)^2 L^2 A^{-2}. \tag{1.1}$$

The constant $(\frac{1}{2}\pi)^2$ is optimal, since equality is attained in the limiting case of an infinite rectangular strip. To obtain these results Makai and Pólya insert in the minimum principle for $\Lambda^2$ functions which depend only on the distance from the boundary.

In this paper we apply a similar method to a two-dimensional membrane $G$ fixed on its exterior bounding curve $C_0$. The membrane is permitted to have interior bounding curves $C_i$ (holes) along which it is free. We shall show that among all such membranes with given area $A$ and given perimeter $L$ of $C_0$ the highest fundamental frequency is attained when $G$ is annular.

This fact gives the upper bound

$$\Lambda \leqslant 2\pi L^{-1} \mu \tag{1.2}$$

where $\mu$ is the lowest root of the transcendental equation

$$J_0(\mu) Y_1(\mu\Psi) = Y_0(\mu) J_1(\mu\Psi) \tag{1.3}$$

with

$$\Psi^2 = 1 - 4\pi A L^{-2}. \tag{1.4}$$

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The classical isoperimetric inequality [1, p. 83] shows that the expression on the right of (1.4) is always nonnegative, and vanishes if and only if $G$ is a circle. The solution of (1.3) is graphed in Jahnke and Emde [3, pp. 207—208]. If $G$ is simply-connected the inequality (1.2) is an improvement of (1.1).

The same method yields an isoperimetric inequality for membranes $G$ which are elastically supported on $C_0$ and free along any inner boundaries $C_i$. The annular membrane has the largest fundamental frequency among all such membranes of given area, perimeter of $C_0$, and elastic constant.

In a similar manner we find a lower bound for the torsional rigidity of a simply connected domain. Again we obtain an improvement of the inequalities of Makai [5,6] and Pólya [7].

The inequalities of Makai and Pólya for the fundamental frequency and torsional rigidity hold for doubly connected (ring-shaped) as well as simply connected domains $G$.

Our bound (1.2) for the fundamental frequency applies when only the outer boundary $C_0$ of $G$ is fixed. However, we may obtain a bound for a membrane $G$ which is fixed along $C_0$ and along one or more inner boundaries $C_i$. To do this, we replace $G$ by a membrane $\bar{G}$ which occupies the same domain and whose boundaries are fixed wherever those of $G$ are fixed, as well as along straight-line paths connecting the fixed boundary components. Then the fundamental frequency $\bar{\Lambda}$ of $\bar{G}$ is greater than $\Lambda$. Moreover, $\bar{G}$ is fixed along a single curve $\bar{C}_0$ consisting of the fixed boundary components of $G$ together with the connecting paths, covered twice. The perimeter $\bar{L}$ of $\bar{C}_0$ exceeds the total length $L$ of the fixed boundary components of $G$ by twice the total length of connecting lines. The area of $\bar{G}$ is again $A$.

Thus, we obtain the bound (1.2) with $L$ replaced by $\bar{L}$ in (1.2) and (1.4). Whether or not this bound is better than (1.1) when $G$ is ring-shaped depends upon the location of the hole.

Similar remarks apply to the torsional rigidity of multiply connected domains.

## II. The Fundamental Frequency

Let $G$ be a plane domain lying inside a simple closed bounding curve $C_0$, and possibly having interior holes bounded by smooth curves $C_i$. Let $\Lambda^2$ be the lowest eigenvalue of the membrane problem:

$$
\begin{aligned}
&\Delta u + \Lambda^2 u = 0 && \text{in } G \\
&u = 0 && \text{on } C_0 \qquad (2.1)\\
&\partial u/\partial n = 0 && \text{on } C_i.
\end{aligned}
$$

It is well known [1, pp. 345—346; 9, p. 87] that

$$\Lambda^2 \leqslant \frac{\iint\limits_G |\operatorname{grad} v|^2 \, dx \, dy}{\iint\limits_G v^2 \, dx \, dy} \tag{2.2}$$

where $v$ is any piecewise continuously differentiable function vanishing on $C_0$.

We define $C_\delta$ to be the curve consisting of points inside $C_0$ at distance $\delta$ from $C_0$. It was shown by Sz.-Nagy [11] that the length $\bar{l}(\delta)$ of $C_\delta$ is well defined for almost all values of $\delta$, and that $\bar{l}(\delta) + 2\pi\delta$ is non-increasing in $\delta$. Thus if $l(\delta)$ is the length of the portion of $C_\delta$ which lies in $G$,

$$l(\delta) \leqslant \bar{l}(\delta) \leqslant L - 2\pi\delta \tag{2.3}$$

where $L = \bar{l}(0)$ is the length of $C_0$.

Let $a(\delta)$ be the area of the portion of $G$ lying between $C_0$ and $C_\delta$. Then

$$a(\delta) = \int_0^\delta l(\delta) \, d\delta. \tag{2.4}$$

Integrating (2.3) gives

$$a(\delta) \leqslant L\delta - \pi\delta^2. \tag{2.5}$$

Inserting (2.3) in this inequality yields

$$\left(\frac{da}{d\delta}\right)^2 = l^2 \leqslant L^2 - 4\pi a(\delta). \tag{2.6}$$

We define a function $r(\delta)$ by

$$4\pi^2 r^2 = L^2 - 4\pi a(\delta). \tag{2.7}$$

If we interpret this equation as a mapping of the portion of $C_\delta$ in $G$ onto the circle of radius $r(\delta)$, we find that $C_0$ is mapped into a circle of equal perimeter and that the portion of $G$ between $C_0$ and $C_\delta$ goes into an annulus of equal area $a(\delta)$. We differentiate (2.7) and use (2.6) and the fact that

$$|\operatorname{grad} \delta| = 1 \tag{2.8}$$

almost everywhere to show that

$$|\text{grad}\, r|^2 \leqslant 1 \tag{2.9}$$

almost everywhere in $G$.

We now let the function $v$ in (2.2) depend only on $r$. In view of (2.9),

$$|\text{grad}\, v|^2 \leqslant \left(\frac{dv}{dr}\right)^2. \tag{2.10}$$

Since the mapping (2.7) is area-preserving, (2.2) becomes

$$\Lambda^2 \leqslant \frac{\int_{r_1}^{r_2} \left(\frac{dv}{dr}\right)^2 r\, dr}{\int_{r_1}^{r_2} v^2 r\, dr}, \tag{2.11}$$

where

$$\begin{aligned} r_1 &= (L^2 - 4\pi A)^{1/2}/2\pi \equiv L\Psi/2\pi, \\ r_2 &= L/2\pi, \end{aligned} \tag{2.12}$$

and $v$ is any differentiable function of $r$ satisfying

$$v(r_2) = 0. \tag{2.13}$$

The right-hand side of (2.11) is the Rayleigh quotient for the annular membrane $\mathring{G}$ whose area is $A$ and whose outer boundary has perimeter $L$. Its minimum under the condition (2.13) is the lowest eigenvalue for the membrane $\mathring{G}$ fixed on the outer boundary and free along the inner boundary. Thus we have established that $\mathring{G}$ has the highest fundamental frequency among all membranes $G$ with given $A$ and $L$.

The minimum value of the expression on the right of (2.11) is attained for

$$v = J_0(2\pi L^{-1}\mu r)Y_0(\mu) - Y_0(2\pi L^{-1}\mu r)J_0(\mu) \tag{2.14}$$

where $\mu$ is determined in such a way that $v'(r_1) = 0$. It is the lowest root of the Eq. (1.3) (cf. [3, pp. 207—208]), and therefore depends upon the dimensionless quantity $\Psi$ defined by (1.4). Substituting (2.14) in (2.11) leads to the bound

$$\Lambda \leqslant 2\pi L^{-1}\mu. \tag{2.15}$$

If $G$ has no holes $C_i$, a lower bound for $\Lambda^2$ in terms of the area $A$ is given by the isoperimetric inequality of Faber [2] and Krahn [4].

$$\Lambda^2 \geqslant \pi j^2 A^{-1}. \tag{2.16}$$

Here $j(\approx 2.4048)$ is the first zero of the Bessel function $J_0$. Equality in (2.16) is attained when $G$ is a circle.

If in (2.11) we choose

$$v = J_0(j[\pi A^{-1}(r^2 - r_1^2)]^{1/2}), \tag{2.17}$$

which satisfies (2.13), we obtain the upper bound

$$\begin{aligned} \Lambda^2 &\leqslant \pi^2 j^2 A^{-1}[1 + (J_1^{-2}(j) - 1)\Psi^2 (1 - \Psi^2)^{-1}] \\ &\leqslant \pi j^2 A^{-1} [1 + 2.712\Psi^2(1 - \Psi^2)^{-1}]. \end{aligned} \tag{2.18}$$

Here $\Psi^2$ is the dimensionless quantity defined by (1.4). Again equality is attained when $G$ is a circle.

The inequalities (2.16) and (2.18) show that if $G$ is simply connected and nearly circular in the sense that $\Psi$ is small, the fundamental frequency $\Lambda$ is near that of the circle of equal area.

Since the function (2.14) yields the best upper bound for $\Lambda^2$, the inequality (2.15) is in general sharper than (2.18).

## III. The Elastically Supported Membrane

We consider the lowest eigenvalue $\Lambda^2(k)$ of the problem

$$\begin{aligned} &\Delta u + \Lambda^2 u = 0 && \text{in } G, \\ &\partial u/\partial n + ku = 0 && \text{on } C_0, \\ &\partial u/\partial n = 0 && \text{on } C_i. \end{aligned} \tag{3.1}$$

The elastic constant $k$ is positive. For any piecewise continuously differentiable function $v$ we have the inequality (cf. [1, pp. 345—346]).

$$\Lambda^2(k) \leqslant \frac{\displaystyle\iint_G |\operatorname{grad} v|^2\, dx\, dy + k \oint_{C_0} v^2\, ds}{\displaystyle\iint_G v^2\, dx\, dy}. \tag{3.2}$$

We introduce the new variable $r$ as in Section II and let $v$ be a function of $r$ only. This gives the upper bound

$$\Lambda^2(k) \leqslant \frac{\int_{r_1}^{r_2} \left(\frac{dv}{dr}\right)^2 r\,dr + 2\pi k r_2 v^2(r_2)}{\int_{r_1}^{r_2} v^2 r\,dr} \tag{3.3}$$

where $r_1$ and $r_2$ are given by (2.12). The right hand side of (3.3) is the Rayleigh quotient for the annular membrane $\mathring{G}$ of area $A$ elastically supported (with elastic constant $k$) on the outer boundary of perimeter $L$, and free on the inner boundary. The minimum of the Rayleigh quotient is the lowest eigenvalue of this membrane. Thus we have shown that $\mathring{G}$ gives the highest fundamental frequency among all membranes $G$ of given $A$, $L$, and $k$. This fact leads to the upper bound

$$\Lambda(k) \leqslant 2\pi L^{-1} \mu \tag{3.4}$$

where $\mu$ is the lowest root of the equation

$$Y_1(\mu\Psi)[kLJ_0(\mu) - 2\pi\mu J_1(\mu)] = J_1(\mu\Psi)[kLY_0(\mu) - 2\pi\mu Y_1(\mu)]. \tag{3.5}$$

If $k$ in problem (3.1) is a nonnegative function of arc length rather than a constant, the inequality (3.4) still holds with $kL$ in (3.5) replaced by $\oint_{C_0} k\,ds$.

## IV. Torsional Rigidity

Let $G$ be a simply connected domain of area $A$ bounded by the closed curve $C_0$ of perimeter $L$. The torsional rigidity $P$ of $G$ is defined by [9, p. 87].

$$P = \max \frac{\left[2\iint_G v\,dx\,dy\right]^2}{\iint_G |\text{grad}\, v|^2\,dx\,dy} \tag{4.1}$$

among sufficiently regular functions $v$ which vanish on $C_0$.

We define the variable $r$ as in section 2 and let

$$v = \frac{1}{2}(r_2^2 - r^2) + r_1^2 \log \frac{r}{r_2}. \tag{4.2}$$

Using the results of Section II leads immediately to the bound

$$P \geqslant \frac{A^2}{2\pi}\,[1 - 2\Psi^2(1-\Psi^2)^{-1} - 4\Psi^4(1-\Psi^2)^{-2}\log\Psi] \tag{4.3}$$

where $\Psi$ is given by (1.4). An upper bound for $P$ in terms of $A$ is given by the isoperimetric inequality

$$P \leqslant A^2/2\pi \tag{4.4}$$

which was conjectured by St. Venant [10] and proved by Pólya [8]. Again we see that if $G$ is nearly circular in the sense that $\Psi$ is small, its torsional rigidity is close to that of the circle of equal area.

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