HANS F. WEINBERGER
1975
Rendiconti di Matematica
(1) Vol. 8, Serie VI.

Invariant sets for weakly coupled parabolic and elliptic systems

by **HANS F. WEINBERGER** (Minneapolis) (*)

To Professor Mauro Picone on his ninetieth birthday.

RIASSUNTO - *Un sistema di m equazioni alle derivate parziali in m funzioni incognite è detto « weakly coupled » se la h-esima equazione contiene solo le derivate parziali della h-esima funzione incognita. Per sistemi semilineari siffatti, di tipo parabolico e di tipo ellittico, vengono determinati insiemi m-dimensionali S (detti insiemi invarianti) tali che se i dati stanno in S, le soluzioni si trovano anche in S.*

1. Introduction.

While a maximum principle for the heat equation was found by E. E. LEVI [8, 9], the systematic study of maximum principles for parabolic equations was inaugurated by Professor PICONE [17, 18]. The strong maximum principle was discovered by L. NIRENBERG [14].

The maximum principle for Laplace's equation is found in the works of GAUSS [6] and EARNSHAW [4]. It was extended to general elliptic equations in the work of PARAF [15], MOUTARD [13], and PICONE [16]. The strong maximum principle is due to HOPF [7].

Two kinds of maximum principles are known for weakly coupled parabolic systems of the form

$$(1.1)\qquad L[u_\alpha] \equiv \frac{\partial u_\alpha}{\partial t} - \sum_{i\ j=1}^{n} a^{ij}(x,t)\frac{\partial^2 u_\alpha}{\partial x_i\,\partial x_j} - \sum_{i=1}^{n} b_i(x,t)\frac{\partial u_\alpha}{\partial x_i} = \sum_{\beta=1}^{m} A_{\alpha\beta}(x,t)\,u_\beta \qquad \alpha = 1, 2, \dots, m.$$

and the corresponding elliptic systems.

(*) This work was supported by the U. S. National Science Foundation through Grant GP 37660 X.

The first kind of maximum principle states that if the off-diagonal entries of the matrix $A_{\alpha\beta}$ are non negative and if the initial and boundary values of the components u_α are nonpositive, then the solution $\mathbf{u}$ also has nonpositive components. Moreover, if $\mathbf{u}(x)$ is a solution of the corresponding elliptic system with the boundary values of the u_α nonpositive, the components u_α are nonpositive. Such results have been found and applied to nonlinear problems by J. SZARSKI [24, 25, 26], W. MLAK [12], J. SCHRÖDER [21, 22], P. BESALA [1, 2], and A. MCNABB [10].

A second kind of maximum principle for the system (1.1) applies if the matrix $A_{\alpha\beta}$ has the property that for any real m-vector ξ

$$\sum_{\alpha,\beta=1}^{m} A_{\alpha\beta}\,\xi_\alpha\,\xi_\beta \leq 0.$$

In this case one finds that if the initial and boundary values of $\mathbf{u}$ satisfy the inequalities $|\mathbf{u}| \leq a$ for some constant a, then the same inequality is satisfied by the solution. Such inequalities were found and used for nonlinear elliptic systems by P. SZEPTYCKI [27] and for linear parabolic systems by T. STYS [23]. The result was extended by C. MIRANDA [11] to certain linear systems in which the first partial derivatives of the u_α are also coupled.

The maximum principle for a single equation states that if the initial and boundary values of $\mathbf{u}$ lie in a certain interval S, the same is true of the solution.

The theorems about the system (1.1) also say that if the initial and boundary values lie in a certain set S, then the same is true of the solution. In the first kind of maximum principle S is taken to be the negative octant while in the second case S is the ball $|\mathbf{u}| \leq a$. A special case of a recent result of SATTINGER [20, Theorem 3.1] is of the same form but with S a certain rectangle.

In this paper we shall show how to find a set S with the property that if the initial and boundary values of a solution $\mathbf{u}$ of the weakly coupled semilinear system

$$L[u_\alpha] = \frac{\partial u_\alpha}{\partial t} - \sum_{i,j=1}^{n} a^{ij}(x,t)\,\frac{\partial^2 u_\alpha}{\partial x_i\,\partial x_j} - \sum_{i=1}^{n} b_i(x,t)\,\frac{\partial u_\alpha}{\partial x_i} = f_\alpha(\mathbf{u}, x, t) \qquad \alpha = 1, 2, \ldots, m. \tag{1.2}$$

lie in S, then the values of $\mathbf{u}$ also lie in S. We call such a set S an *invariant set*.

Theorem 1 shows that under some regularity conditions a set S which is convex and which has the property that $\mathfrak{f}$ never points outward on the boundary of S is an invariant set for the system (1.2).

For a scalar equation the set S is, of course an interval. The strong maximum principle then states that if the value of the solution at an interior point lies on the boundary of S, then all the values of the solution lie on the boundary of S. In Theorem 2 we prove the analogous theorem for our invariant sets.

In Section 3 we give the corresponding results for the weakly coupled elliptic system which is obtained from (1.2) by making the coefficients, the vector field $\mathfrak{f}$, and the solution independent of t.

In Section 4 we show that our theorems may be sharpened by introducing additional dependent variables whose values are x and t.

In Section 5 we present some examples to show that our sufficient conditions for S to be invariant are in some sense also necessary. We also give some examples to illustrate the application of our results.

It should be noted that while the first kind of maximum principle also works when the operators on the left of (1.1) depend on α, our results, like the second kind of maximum principle, do not extend to this case. Our results also do not contain the above-mentioned generalization of C. Miranda to elliptic systems with coupled derivatives.

I wish to thank my colleagues Lawrence Markus and George Sell for helpful discussions.

2. Parabolic systems.

We beging with a simple lemma which contains the basic idea of our results.

We denote by D a domain in R^n with closure $\bar{D}$.

If S is a closed convex set in R^m, we shall denote by S_ρ $(\rho>0)$ the parallel set of all points in R^m whose euclidean distance to S is at most ρ.

Lemma 1. *Let S be a closed convex subset of R^m. Suppose that there is an $\varepsilon>0$ such that for each $\rho\in(0,\varepsilon)$ the parallel set S_ρ has the property that if $\mathfrak{p}$ is the outward normal at a point $\mathfrak{u}^*$ on the boundary of S_ρ, then*

$$\mathfrak{p}\cdot\mathfrak{f}(\mathfrak{u}^*, x, t) < 0 \tag{2.1}$$

for all (x, t) in $D\times(0, T]$.

If $\mathbf{u}$ *is any solution of the parabolic system* (1.2) *in* $D\times(0,T]$ *and if all limit points of* $\mathbf{u}(x,t)$ *as* (x,t) *approaches* $\bar{D}\times\{0\}$ *or* $\partial D\times\times[0,T]$ *and, if* D *is unbounded, as* $|x|\to\infty$ *with* $(x,t)\in D\times(0,T]$ *lie in* S, *then* $\mathbf{u}(x,t)\in S$ *in* $D\times(0,T]$.

PROOF. Suppose that for some $(x,t)\in D\times(0,T]$, $\mathbf{u}(x,t)$ is outside S. Then $\mathbf{u}(x,t)$ is also outside S_ϱ for some $\rho\in(0,\varepsilon)$. By continuity there is a point $(x^*,t^*)\in D\times(0,T]$ such that $\mathbf{u}(x,t)\in S_\varrho$ for $(x,t)\in\in D\times(0,t^*]$ and $\mathbf{u}^*\equiv\mathbf{u}(x^*,t^*)\in\partial S_\varrho$. Let $\mathbf{p}$ be the outward normal to ∂S_ϱ at $\mathbf{u}^*$. Then by hypothesis $\mathbf{p}\cdot\mathbf{f}(\mathbf{u}^*,x^*,t^*)<0$ On the other hand, since S_ϱ is convex, the function $\mathbf{p}\cdot\mathbf{u}(x,t)$ attains its maximum value in $D\times(0,t^*]$ at (x^*,t^*). Therefore at (x^*,t^*)

$$\mathbf{p}\cdot\partial\mathbf{u}/\partial t\geq 0,\quad \mathbf{p}\cdot\partial\mathbf{u}/\partial x_i=0\quad\text{for}\quad i=1,\dots,n,$$

and the matrix $\mathbf{p}\cdot\partial^2\mathbf{u}/\partial x_i\,\partial x_j$ is negative semidefinite. A standard argument shows that $\mathbf{p}\cdot\Sigma\, a^{ij}\,\partial^2\mathbf{u}/\partial x_i\,\partial x_j$ is nonpositive. Thus it follows from (1.2) that $\mathbf{p}\cdot\mathbf{f}(\mathbf{u}^*,x^*,t^*)\geq 0$, which contradicts (2.1). We conclude that $\mathbf{u}(x,t)$ cannot lie outside S for $(x,t)\in D\times(0,T]$, which proves the lemma.

If we make some smoothness assumptions about the system (1.2) and the domain D, we can replace the conditions (2.1) on the boundaries of a whole family of parallel sets by the weaker condition

$$\mathbf{p}\cdot\mathbf{f}(\mathbf{u}^*,x,t)\leq 0\quad\text{for}\quad(x,t)\in D\times(0,T] \tag{2.2}$$

on the boundary of S.

It can be seen from the work of EIDEL'MAN [5, especially Theorem 4.4] that if

(*a*) L is uniformly parabolic and its coefficients are uniformly Hölder continuous with Hölder exponent greater then 1/2, and

(*b*) the boundary ∂D is of class $C^{1,\nu}$, $\nu\in(0,1)$ and the vector field $\mathbf{f}$ is uniformly Hölder continuous in x and t and Lipschitz continuous in $\mathbf{u}$ for $(x,t)\in D\times(0,T]$, then for any bounded Hölder continuous initial values $\mathbf{u}(x,0)$ in $\bar{D}$ and boundary values $\mathbf{u}$ on $\partial D\times[0,T]$ the system (1.2) has a unique bounded solution in $D\times(0,T]$ which is continuous in $\bar{D}\times[0,T]$. Moreover, if D is unbounded, this solution is the limit of the solutions $\mathbf{u}_R$ *of* initial-boundary value problems on $D\cap B_R\times(0,T]$ with $\mathbf{u}_R=\mathbf{u}$ at $t=0$ and on $\partial D\cap B_R\times(0,T]$ and with any uniformly bounded smooth data given on $D\cap\partial B_R\times(0,T]$. B_R denotes the ball $\{x:|x|<R\}$.

THEOREM 1. *Let S be a closed convex subset of R^m with the property that for any outward normal $\mathfrak{p}$ at any boundary point $\mathfrak{u}^*$ of S the inequality (2.2) is satisfied.*

Let the operator L satisfy the above hypothesis (a), and let D and $\mathbf{f}$ satisfy hypothesis (b).

If $\mathfrak{u}(x, t)$ is any solution of (1.2) in $D\times(0, T]$ which is continuous in $\bar{D} \times [0, T]$, and if the values of $\mathfrak{u}$ on $D\times\{0\}$ and on $\partial D\times[0,T]$ are bounded and Hölder continuous and lie in S, then $\mathfrak{u}(x, t)\in S$ in $D\times(0, T]$.

PROOF. For any point $\mathfrak{w}$ of R^m which lies outside S there is a unique point $\mathfrak{q}(\mathfrak{w})$ on ∂S which is closest to $\mathfrak{w}$. Of course, $\mathfrak{w}-\mathfrak{q}(\mathfrak{w})$ is an outward normal to ∂S at $\mathfrak{q}(\mathfrak{w})$. We define

$$\mathbf{g}(\mathfrak{w}, x, t) = \begin{cases} \mathbf{f}(\mathfrak{w}, x, t) & \text{if } \mathfrak{w} \in S \\ \mathbf{f}(\mathfrak{q}(\mathfrak{w}), x, t) - \mathfrak{w} + \mathfrak{q}(\mathfrak{w}) & \text{if } \mathfrak{w} \notin S. \end{cases}$$

The vector field $\mathbf{g}$ is Hölder continuous in x and t and Lipschitz continuous in $\mathfrak{w}$. Assume first that D is bounded. By hypothesis there is a unique solution $\mathfrak{w}$ of the problem

$$L\mathfrak{w} = \mathbf{g}(\mathfrak{w}, x, t) \text{ in } D\times(0, T], \;\; \mathfrak{w}(x,0) = \mathfrak{u}(x,0), \;\; \mathfrak{w}(x,t)\,|_{\partial D} = \mathfrak{u}(x, t)\,|_{\partial D}.$$

It is easily seen that if $\mathfrak{w}^*$ is a boundary point of the parallel set S_ϱ, then $\mathfrak{p}=\mathfrak{w}^*-\mathfrak{q}(\mathfrak{w}^*)$ is an outward normal vector to ∂S_ϱ at $\mathfrak{u}^*$ and to ∂S at $\mathfrak{q}(\mathfrak{w}^*)$. Hence

$$\mathfrak{p}\cdot\mathbf{g}(\mathfrak{w}^*, x, t) = (\mathfrak{w}^* - \mathfrak{q}(\mathfrak{w}^*))\cdot\mathbf{f}(\mathfrak{w}^*, x, t) - |\,\mathfrak{w}^* - \mathfrak{q}(\mathfrak{w}^*)\,|^2 \leq -\varrho^2$$

because of (2.2). Therefore the parallel sets of S satisfy (2.1) with respect to the vector field $\mathbf{g}$. Then Lemma 1 shows that $\mathfrak{w}(x, t)\in S$ in $D\times(0, T]$.

If D is unbounded, $\mathfrak{w}$ is the limit of solutions $\mathfrak{w}_R$ in $D\cap B_R\times(0,T]$, where $\mathfrak{w}_R=\mathfrak{u}$ on $D\cap B_R\times\{0\}$ and on $\partial D\cap B_R\times[0, T]$, and $\mathfrak{w}_R$ on $D\cap\partial B_R\times(0, T]$ is taken to be smooth and uniformly bounded and to lie in S. By the above argument each $\mathfrak{w}_R$ lies in S. Since S is closed, the limit $\mathfrak{w}$ lies in S.

Since $\mathbf{g}(\mathfrak{u}, x, t)\equiv\mathbf{f}(\mathfrak{u}, x, t)$ for $\mathfrak{u}\in S$, $\mathfrak{w}$ is a bounded solution of the system (1.2) with the same initial and boundary data as $\mathfrak{u}$. A standard uniqueness theorem shows that $\mathfrak{u}\equiv\mathfrak{w}$. Hence $\mathfrak{u}(x,t)\in S$ in $D\times(0,T]$, and the theorem is proved.

Note that if $\mathbf{f}$ is independent of x and t, the condition (2.2) means that the set S is invariant with respect to increasing t under the flow $d\,\mathbf{u}/dt = \mathbf{f}(\mathbf{u})$. This system is obtained from (1.2) by deleting the terms involving second partial derivatives.

This system is obtained from (1.2) by deleting the terms involving second partial derivatives.

In order to formulate a strong maximum principle we shall need the following definition.

DEFINITION. *The boundary point* $\mathbf{u}^*$ *of* S *is said to satisfy a* **slab condition** *if there is a neighborhood* N *of* $\mathbf{u}^*$ *in* R^m *with the following property.*

There is a C^1 *one-to-one mapping* M *with* C^1 *inverse of* $N \cap S$ *onto a set of the form*

$$\}\ \tau_1, \tau_2, \dots, \tau_{m-1}, \sigma : (\tau_1, \dots, \tau_{m-1}) \in K,\ \ 0 \leq \sigma \leq \mu(\tau_1, \dots, \tau_{m-1})\ \}$$

such that

(a) $M\mathbf{u}^* \in K \times \{0\}$,

(b) $M\mathbf{u} \in K \times \{0\} \Longrightarrow \mathbf{u} \in \partial S$

(c) The component $\sigma(\mathbf{u})$ *of* M *is a* C^2 *function of* $\mathbf{u}$.

(d) The level surfaces $\sigma =$ constant *are convex.*

In the following theorem D is an arbitrary domain in R^n.

THEOREM 2. *Let* S *be a closed convex subset of* R^m *such that every boundary point of* S *satisfies a slab condition. Let* $\mathbf{f}(\mathbf{u}, x, t)$ *be Lipschitz continuous in* $\mathbf{u}$, *and suppose that if* $\mathbf{p}$ *is any outward normal at a boundary point* $\mathbf{u}^*$, *then* (2.2) *holds for all* $(x, t) \in D \times (0, T]$.

Let L *be locally uniformly parabolic, and let its coefficients be locally bounded.*

If $\mathbf{u}$ *is any solution in* $D \times (0, T]$ *of the system* (1.2) *with* $\mathbf{u}(x, t) \in S$ *and if* $\mathbf{u}(x^*, t^*) \in \partial S$ *for some* $(x^*, t^*) \in D \times (0, T]$, *then* $\mathbf{u}(x, t) \in \partial S$ *in* $D \times (0, t^*]$.

PROOF. Let N be the neighborhood of $\mathbf{u}^* \equiv \mathbf{u}(x^*, t^*)$ in which the mapping M is defined. By continuity there is a subdomain D_1 of D and a $t_1 \in (0, t^*)$ such that $\mathbf{u}(x, t) \in N$ for $(x, t) \in D_1 \times (t_1, t^*]$. Denote by $\varphi(x, t)$ the σ-component $\sigma(\mathbf{u}(x, t))$ of M applied to $\mathbf{u}(x, t)$ in $D_1 \times (t_1, t^*]$.

We see from the chain rule and (1.2) that

$$L[\varphi]=\sum_{\alpha=1}^{m}\frac{\partial\sigma}{\partial u_\alpha}(\mathfrak{u})f_\alpha(\mathfrak{u},x,t)-\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{m}a^{ij}\frac{\partial^2\sigma}{\partial u_\alpha\,\partial u_\beta}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\beta}{\partial x_j},\tag{2.3}$$

where $\mathfrak{u}$ is, of course, to be evaluated at (x, t).

Since the mapping M gives the τ_μ as functions of $\mathfrak{u}$, we may write

$$\tau_\mu(x,t)\equiv\tau_\mu(\mathfrak{u}(x,t)).$$

The inverse mapping M^{-1} allows us to write $\mathfrak{u}$ as a function of $\tau_1, \dots, \tau_{m-1}$ and σ. Then

$$\frac{\partial u_\alpha}{\partial x_i}=\sum_{\nu=1}^{m-1}\frac{\partial u_\alpha}{\partial\tau_\nu}\frac{\partial\tau_\nu}{\partial x_i}+\frac{\partial u_\alpha}{\partial\sigma}\frac{\partial\varphi}{\partial x_i}.$$

Because the surfaces $\sigma=$ constant are convex, the matrix

$$\sum_{\alpha,\beta=1}^{m}\frac{\partial^2\sigma}{\partial u_\alpha\,\partial u_\beta}\frac{\partial u_\alpha}{\partial\tau_\nu}\frac{\partial u_\beta}{\partial\tau_\mu}\tag{2.4}$$

is negative semidefinite. Hence

$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{m}\sum_{\nu,\mu=1}^{m-1}a^{ij}\frac{\partial^2\sigma}{\partial u_\alpha\,\partial u_\beta}\frac{\partial u_\alpha}{\partial\tau_\nu}\frac{\partial u_\beta}{\partial\tau_\mu}\frac{\partial\tau_\nu}{\partial x_i}\frac{\partial\tau_\mu}{\partial x_j}\leq 0.$$

We then see from (2.3) that

$$\begin{aligned}\tilde{L}[\varphi]\equiv L[\varphi]+\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{m}a^{ij}\frac{\partial^2\sigma}{\partial u_\alpha\,\partial u_\beta}&\left(2\sum_{\nu=1}^{m-1}\frac{\partial u_\alpha}{\partial\tau_\nu}\frac{\partial\tau_\nu}{\partial x_i}+\frac{\partial u_\alpha}{\partial\sigma}\frac{\partial\varphi}{\partial x_i}\right)\cdot\\ \cdot\frac{\partial u_\beta}{\partial\sigma}\frac{\partial\varphi}{\partial x_j}&\geq\sum\frac{\partial\sigma}{\partial u_\alpha}(\mathfrak{u})f_\alpha(\mathfrak{u},x,t)\equiv F(\mathfrak{u},x,t).\end{aligned}\tag{2.5}$$

We again consider $\mathfrak{u}$ as a function of τ and σ which, in turn, are functions of x and t. We define the function

$$H(\sigma,x,t)\equiv\begin{cases}F(\mathfrak{u}(\tau(x,t),0),x,t)+\\ \quad+\dfrac{\sigma}{\varphi(x,t)}\left[F(\mathfrak{u}(\tau(x,t),\varphi(x,t)),x,t)-F(\mathfrak{u}(\tau(x,t),0),x,t)\right]\\ \qquad\qquad\text{for } 0\leq\sigma<\varphi(x,t)\\ F(\mathfrak{u}(\tau(x,t),\varphi(x,t)),x,t)\qquad\text{for } \sigma\geq\varphi(x,t).\end{cases}$$

Then (2.5) shows that φ satisfies the uniformly parabolic differential inequality

$$\widetilde{L}[\varphi] \geq H(\varphi, x, t) \tag{2.6}$$

in $D_1 \times (t_1, t^*]$.

Since the points where $\sigma = 0$ lie on the boundary of S and since $\partial\sigma/\partial u_\alpha$ is an inward normal at such a point, it follows from (2.2) that

$$H(0, x, t) = \Sigma \frac{\partial \sigma(\mathbf{u})}{\partial u_\alpha} f_\alpha(\mathbf{u}, x, t) \geq 0 .$$

Thus (2.6) implies that

$$\widetilde{L}[\varphi] \geq H(\varphi, x, t) - H(0, x, t). \tag{2.7}$$

Moreover, $\varphi \geq 0$ in $D_1 \times (t_1, t^*]$ and $\varphi(x^*, t^*) = 0$.

By construction $H(\sigma, x, t)$ is Lipschitz continuous in σ. Since φ is bounded in any closed subset $\overline{D}_2 \times [t_2, t^*]$ of $D_1 \times (t_1, t^*]$, there is a constant c so that $|H(\varphi(x, t), x, t) - H(0, x, t)| \leq c\,|\varphi(x, t)|$ for $(x, t) \in D_2 \times [t_2, t]$. Hence if we define

$$q(x, t) = e^{-ct}\varphi(x, t)$$

and

$$\mu(x, t) \equiv \begin{cases} e^{-ct}\left[\dfrac{H(\varphi(x, t), x, t) - H(0, x, t)}{\varphi(x, t)} - c\right] & \text{if } \varphi(x, t) \neq 0 \\ 0 & \text{if } \varphi(x, t) = 0, \end{cases}$$

we see that $\mu \leq 0$ and $q \geq 0$. Hence, $\widetilde{L}q \geq \widetilde{L}q - \mu q \geq 0$, $q \geq 0$ in $\overline{D}_2 \times \times [t_2, t^*]$, and $q(x^*, t^*) = 0$. It follows from the strong maximum principle (see, e. g. [19, Theorem 2. 5]) that $q \equiv 0$ and hence that $\varphi \equiv 0$ in $D_2 \times [t_1, t^*]$. Since $\overline{D}_2 \times [t_2, t^*]$ is an arbitrary closed subcylinder of $D_1 \times (t_1, t^*]$, $\varphi \equiv 0$ in $D_1 \times (t_1, t^*]$. This means that $\mathbf{u}(x, t) \in \partial S$ in this set.

The above argument shows that the set $\{(x, t) \in D \times (0, t^*] : \mathbf{u}(x,t) \in \in \partial S\}$ is relatively open in $D \times (0, t^*]$. By continuity, this set is also relatively closed. We conclude that $\mathbf{u}(x, t) \in \partial S$ in all of $D \times (0, t^*]$.

REMARKS.

1. The slab condition is certainly satisfied at each boundary point if the boundary ∂S is of class C^2. It is also satisfied if S is the intersec-

tion of finitely many convex sets with C^2 boundaries, provided the boundaries intersect at nonzero angles.

2. If $\mathbf{u}^*$ lies on the intersection of several C^2 boundary components which meet at nonzero angles, the proof of Theorem 2 applied to each piece separately shows that if $\mathbf{u}(x^*, t^*)=\mathbf{u}^*$, then $\mathbf{u}(x, t)$ lies on the intersection of these components in $D\times(0, t^*]$. Thus if we think of ∂S as a curvilinear polyhedron, $\mathbf{u}$ is confined to one of the faces, edges, or vertices of S.

3. If the Gaussian curvature of ∂S is never zero, then the matrix (2.4) is negative definite. Since $\varphi\equiv 0$, the inequality in (2.7) cannot be strict. Therefore $\partial\tau_\nu/\partial x_i=0$ for all ν and i at any point (x^*, t^*) where $\mathbf{u}(x^*, t^*)\in\partial S$. Since also $\partial\varphi/\partial x_i=0$ at such a point, we conclude that in this case a solution $\mathbf{u}(x, t)$ which lies on ∂S must be independent of x.

4. The above proof is easily extended to show that if (1.2) holds in an arbitrary domain R of the (x, t) space, if $\mathbf{u}(x, t)\in S$ in R, and if $\mathbf{u}(x^*, t^*)\in\partial S$, then $\mathbf{u}(x, t)\in\partial S$ at all points which can be reached from (x^*, t^*) by a path in R along which t never increases.

3. Elliptic systems.

If the coefficients and the function $\mathbf{f}$ in (1.2) are independent of t, we may consider the corresponding uniformly elliptic system

$$(3.1)\quad M[u_\alpha]\equiv -\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u_\alpha}{\partial x_i\,\partial x_j}-\sum_{i=1}^{n} b_i(x)\frac{\partial u_\alpha}{\partial x_i}=f_\alpha(\mathbf{u}, x)\quad \alpha=1,\dots,m.$$

By using essentially the arguments that were used to prove Theorem 1, one establishes the following theorem about a solution $\mathbf{u}$ of (3.1) in a domain D.

THEOREM 3. *Let S be a closed convex subset of R^m with the property that if $\mathbf{p}$ is any outward normal to ∂S at a boundary point $\mathbf{u}^*$, then*

$$\mathbf{p}\cdot\mathbf{f}(\mathbf{u}^*, x)\leq 0 \quad\text{for}\quad x\in D.$$

Suppose that any boundary value problem for equation (3.1) with boundary values in S has at most one solution, and that such a boundary value problem with $\mathbf{f}$ replaced by the vector field $\mathbf{g}$ defined in the proof of Theorem 1 has at least one solution.

Let $\mathfrak{u}$ *be a solution of the system* (3.1), *and let its boundary values lie in S. Then* $\mathfrak{u}(x) \in S$ *for all x in D.*

Since a solution of (3.1) is also a solution of the corresponding parabolic system (1.1), Theorem 2 yields the following result.

THEOREM 4. *Let S be a closed convex subset of* R^m, *and let every point of the boundary satisfy a slab condition. Let* $\mathfrak{f}(\mathfrak{u}, x)$ *be Lipschitz continuous in* $\mathfrak{u}$, *and suppose that any outward normal* $\mathfrak{p}$ *at any boundary point* $\mathfrak{u}^*$ *of S satisfies the inequality* $\mathfrak{p}\cdot\mathfrak{f}(\mathfrak{u}^*, x) \leq 0$ *for all x in D.*

If $\mathfrak{u}(x, t)$ *is a solution of the elliptic system* (3.1), *if* $\mathfrak{u}(x, t) \in S$, *in D, and if* $\mathfrak{u}(x^*, t^*) \in \partial S$ *for some point* $(x^*, t^*) \in D$, *then* $\mathfrak{u}(x, t) \in \partial S$ *for all* $(x, t) \in D$.

The first three remarks after Theorem 2 apply here. In particular Remark 3 states that if $\mathfrak{u}$ lies on a face of ∂S whose Gaussian curvature does not vanish, the $\mathfrak{u}$ is constant.

4. Finer invariant sets.

If the vector field $\mathfrak{f}(\mathfrak{u}, x, t)$ is independent of x and t, we can easily describe the invariant sets S. They are those closed convex sets which are invariant with respect to increasing t under the flow $d\mathfrak{u}/dt = \mathfrak{f}(\mathfrak{u})$. This is the system of equations which one obtains by eliminating the second order terms from the system (1.2).

We can always reduce the system (1.2) to a system in which $\mathfrak{f}$ depends only on the dependent variable $\mathfrak{u}$ by introducing the new variables $u_{m+1}, \dots, u_{m+n}, u_{m+n+1}$ in such a way that $u_{m+i} = x_i$ for $i = 1, \dots, n$ and $u_{m+n+1} = t$. We simply append the additional equations

$$(4.1) \quad L[u_\alpha] = -b_{\alpha-m} \text{ for } \alpha = m+1, \dots, m+n, \quad L[u_{m+n+1}] = 1$$

with the proper initial and boundary values. We let $\hat{\mathfrak{u}}$ be the new $m+n+1$-dimensional vector of unknowns and write the combination of the systems (1.2) and (4.1) as

$$(4.2) \quad L[\hat{\mathfrak{u}}] = \hat{\mathfrak{f}}(\hat{\mathfrak{u}}).$$

Theorem 1 applied to this new system now gives invariant sets $\hat{S}$ in R^{m+n+1}. Since we know that (x, t) remains in $D \times (0, T]$, we only need to define $\hat{S}$ as a subset of $R^m \times D \times (0, T]$. The only boundary points of interest are those of the relative boundary in $R^m \times D \times (0, T]$. The proof of Theorem 1 gives the following extension.

THEOREM 5. *Let the domain D, and the operator L satisfy the hypotheses of Theorem 1. Let $b_1(x, t), \dots, b_n(x, t)$ and $\mathbf{f}(\mathbf{u}, x, t)$ be Lipschitz continuous in all their variables.*

Let $\hat{S}$ be a relatively closed subset of $R^m \times D \times (0, T]$. Suppose that each point $\hat{\mathbf{u}}^*$ of the relative boundary $\hat{B}$ of $\hat{S}$ in $R^m \times D \times (0, T]$ has a neighborhood whose intersection with $\hat{S}$ is convex and that any outward normal $\hat{\mathbf{p}}$ at $\hat{\mathbf{u}}^*$ satisfies the inequality

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{f}}(\hat{\mathbf{u}}^*) \leq 0. \tag{4.3}$$

If $\hat{\mathbf{u}}$ is a solution in $D \times (0, T]$ of (4.2) whose initial and boundary values lie in $\hat{S}$, then $\hat{\mathbf{u}} \in \hat{S}$ in $D \times (0, T]$.

We remark that if $S \subset R^m$ satisfies the conditions of Theorem 1, then the set $\hat{S} = S \times D \times (0, T]$ satisfies the conditions of Theorem 5.

The set $\hat{S} \cap R^m \times \{(x, t)\}$ in which $\mathbf{u}(x, t)$ may lie may depend upon the point (x, t). Therefore Theorem 5 may give more information than Theorem 1 even if the vector field $\mathbf{f}$ is independent of x and t.

The reduction to the system (4.2) can also be used to extend the other theorems, but we shall not do so here.

We note that by (4.3) a subset $\hat{S}$ of $R^m \times D \times (0, T]$ is an invariant set for (4.2) if it is locally convex on its relative boundary and invariant with respect to increasing time under the flow

$$d\mathbf{u}/dt = \mathbf{f}(\mathbf{u}, x, t); \quad dx_i/dt = -b_i(x, t), \quad i = 1, \dots, n. \tag{4.4}$$

These are the equations which can be used to solve the system which is obtained from (4.2) by discarding the second derivative terms of L (that is, the diffusion terms).

5. Examples and counterexamples.

In this section we shall give some examples to illustrate the implications of our results.

The requirement of convexity makes it difficult to find invariant sets S. The first example shows that the convexity is needed.

EXAMPLE 1. *Let $m=2$, $n=1$, and consider the following problem*

$$\frac{\partial u_1}{\partial t}-\frac{\partial^2 u_1}{\partial x^2}=0,\quad \frac{\partial u_2}{\partial t}-\frac{\partial^2 u_2}{\partial x^2}=0 \tag{5.1}$$

$$u_1(x,0)=u_2(-x,0)=\begin{cases} 0 & \text{for } x\leq 0 \\ x & \text{for } 0<x<1 \\ 1 & \text{for } x\geq 1. \end{cases}$$

Since $\mathbf{f}\equiv 0$, the condition $\mathbf{p}\cdot\mathbf{f}\leq 0$ is satisfied for any set S. The initial values lie on the union of the two line segments $u_1=0$, $u_2\in[0,1]$ and $u_2=0$, $u_1\in[0,1]$. By the maximum principle $u_1(x,t)\in(0,1)$, $u_2(x,t)\in$ $\in(0,1)$, and $u_1(x,t)+u_2(x,t)\in(0,1)$. Moreover, for each t

$$\lim_{x\to\infty} u_1(x,t)=\lim_{x\to-\infty} u_2(x,t)=1 \quad\text{and}\quad \lim_{x\to-\infty} u_1(x,t)=\lim_{x\to-\infty} u_2(x,t)=0.$$

Finally, since $1-u_1(x,0)-u_2(x,0)$ is bounded and has bounded support, the explicit solution of the heat equation shows that $1-u_1(x,t)-$ $-u_2(x,t)\leq ct^{-1/2}$. Thus, the solution comes arbitrarily close to the line $u_1+u_3=1$. It follows from continuity that the range of $\mathbf{u}(x,t)$ in $R\times R_+$ is just the triangle $u_1>0$, $u_2>0$, $u_1+u_2<1$, whose closure is the convex hull of the set of initial values. Thus the smallest closed invariant set S is this convex hull.

We remark that in this particular problem, the smallest relatively closed invariant set S given by Theorem 5 is just $S\times R^1\times(0,T]$.

The following example shows that under the conditions of Theorem 2 $\mathbf{u}$ need not be constant.

EXAMPLE 2. The disc $S=\{\mathbf{u}:|\mathbf{u}|^2\leq 1\}$ satisfies the conditions of Theorem 2 with respect to the system

$$\frac{\partial u_1}{\partial t}-\frac{\partial^2 u_1}{\partial x^2}=u_2,\quad \frac{\partial u_2}{\partial t}-\frac{\partial^2 u_2}{\partial x^2}=-u_1.$$

This system has the noncostant solution $u_1=\sin t$, $u_2=\cos t$, which lies on ∂S for all t.

We shall now give an example to illustrate the fact that the nonzero Gauss curvature is needed in Remark 3 after Theorem 2.

EXAMPLE 3. The cylinder $S=\{\mathfrak{u}: u_1^2+u_2^2=1,\ u_3\in[-1,1]\}$ satisfies the conditions of Theorem 2 with respect to the system

$$\frac{\partial u_1}{\partial t}-\frac{\partial^2 u_1}{\partial x^2}=u_2\,,\qquad \frac{\partial u_2}{\partial t}-\frac{\partial^2 u_2}{\partial x^2}=-u_1,\qquad \frac{\partial u_3}{\partial t}-\frac{\partial^2 u_3}{\partial x^2}=0\,.$$

The Gaussian curvature vanishes on the side as well as on the top and bottom of this cylinder. A solution on ∂S is given by $u_1=\sin t$, $u_2=\cos t$, and $u_3(x,t)$ any solution of the heat equation with values in $(-1,1)$. Thus $\mathfrak{u}$ need not be independent of x.

We shall now give some examples of invariant sets S in the case when $\mathbf{f}(\mathfrak{u},x,t)\equiv A\mathfrak{u}$, where A is a constant $m\times m$ matrix. We recall that a closed convex set S is an invariant set if and only if it is invariant with respect to increasing t under the flow.

$$d\mathfrak{u}/dt=A\mathfrak{u}. \tag{5.2}$$

Thus, if S contains an initial point of a trajectory of (5.2), it must contain all points of the trajectory with $t>0$.

It is known [3, § 56, Theo. 2] that the flow (5.2) has bounded invariant sets with interior if and only if all the eigenvalues of A have nonnegative real parts and no nonlinear elementary divisors correspond to any eigenvalue with real part zero. Let S be the matrix such that $S^{-1}AS$ is the Jordan canonical form but with the usual ones above the diagonal of each Jordan block replaced by the real part of the corresponding eigenvalue. It is easily verified that the positive definite matrix $R=\operatorname{Re} S^{*-1}S^{-1}$ has the property that $\mathfrak{u}\cdot RA\mathfrak{u}\leq 0$ for all $\mathfrak{u}$. Therefore $\mathfrak{u}\cdot RA\mathfrak{u}$ is a Lyapounov function for (5.2). Moreover, it follows that if $L[\mathfrak{u}]=A\mathfrak{u}$, then $L[\mathfrak{u}\cdot R\mathfrak{u}]\leq 0$.

The ellipsoids $\mathfrak{u}\cdot R\mathfrak{u}\leq a^2$ are therefore invariant sets S.

If the b_i are all zero, Theorem 5 shows that if $\psi(x)$ is any concave function, the set $\hat{S}=\{(\mathfrak{u},x,t): \mathfrak{u}\cdot R\mathfrak{u}\leq\psi(x)^2,\ (x,t)\in D\times(0,T]\}$ is an invariant set in R^{m+n+1}. Thus if $\mathfrak{u}(x,0)\cdot R\mathfrak{u}(x,0)\leq\psi(x)^2$ on $\bar{D}\times\{0\}$ and on $\partial D\times[0,T]$, then $\mathfrak{u}\cdot R\mathfrak{u}\leq\psi(x)^2$ in $D\times(0,T]$.

One may prefer to use sets S other than ellipsoids. For example, if

$$A=\begin{pmatrix}-10 & -1\\ 10 & -1\end{pmatrix},$$

any rectangle of the form $|u_1|\leq a$, $|u_2|\leq 10\,a$ is an invariant set. Moreover, if the b_i are all zero and if $\psi(x)$ is any concave function of x, the set $\hat{S}=R^m\times D\times(0,T]\cap\{u,x,t)\colon |u_1|\le\psi(x),\ \ |u_2|\le 10\psi(x)\}$ is invariant.

Even if there are no bounded invariant sets with interior, one can frequently find unbounded invariant sets, and these can also be useful. For instance, if $m=2$ and

$$A=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$$

the origin is a saddle point for the flow (5.2). It is easily seen that any strip $u_1\ge a,\ b\le u_2\le c$ with $a\ge 0,\ b\le 0$ and $c\ge 0$, and any strip $u_1\le a,\ b\le u_2\le c$ with $a\le 0,\ b\le 0$ and $c\ge 0$ is invariant.

In fact, one can always find such invariant sets when $m=2$ except in the case when A has complex eigenvalues with positive real part. In this case the origin is an unstable spiral point with respect to the flow (5.2), and hence the convex hull of any trajectory is the whole plane. Thus, the only invariant sets in this case are the origin and the whole plane.

BIBLIOGRAPHY

[1] P. Besala: *On solutions of Fourier's first problem for a system of non-linear parabolic equations in an unbounded domain.* Ann. Polon. Math. 13 (1963) pp. 247-265.

[2] P. Besala: *Concerning solutions of an exterior boundary-value problem for a system of non-linear parabolic equations.* Ann. Polon. Math. 14 (1964) pp. 289-301.

[3] W. A. Coppel: *Stability and Asymptotic Behavior of Differential Equations.* D. C. Heath, Boston, 1965.

[4] S. Earnshaw: *On the nature of the molecular forces which regulate the constitution of the luminiferous ether.* Cambridge Phil. Soc. Trans. 7 (1839) pp. 97-112.

[5] S. D. Eidel'man: *Parabolic Systems.* North-Holland, Amsterdam, 1969.

[6] C. F. Gauss: *Allgemeine Theorie des Erdmagnetismus.* Beobachten des magnetischen Vereins im Jahre 1838, Leipzig, 1839. Also in Werke (Collected Works) 5, p. 129.

[7] E. Hopf: *Elementare Bermerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordung vom elliptischen Typus.* Sitzungsber. der Preuss. Akad. der Wissenschaften 19 (1927) pp. 147-152.

[8] E. E. Levi: *Sull'equazione del calore.* Reale Accad. dei Lincei, Roma. Rendiconti (5) 162 (1907) pp. 450-456.

[9] E. E. Levi: *Sull'equazione del calore.* Annali di Mat. Pura ed Appl. 14 (1908) pp. 187-264.

[10] A. McNabb: *Comparison and existence theorems for multicomponent diffusion systems.* J. of Math. Anal. and Applic. 3 (1961) pp. 133-144.

[11] C. Miranda: *Sul teorema del massimo modulo per una classe di sistemi ellittici di equazioni del secondo ordine e per le equazioni a coefficienti complessi.* Istituto Lombardo-Accademia di Scienze e Lettere. Rendiconti (A) 104 (1970) pp. 736-745.

[12] W. Mlak: *Differential inequalities of parabolic type.* Ann. Polon. Math. 3 (1957) pp. 349-354.

[13] T. Moutard: *Notes sur les équations aux dérivées partielles.* J. de l'École Polytechnique 64 (1894) pp. 55-69.

[14] L. Nirenberg: *A strong maximum principle for parabolic equations* Comm. on Pure and Applied Math. 6 (1953) pp. 167-177.

[15] A. PARAF: *Sur le problème de Dirichlet et son extension au cas de l'équation linéaire générale du second ordre.* Annales de la Faculté des Sciences de Toulouse (I) 6 (1892) H, pp. 1-75.

[16] M. PICONE: *Maggiorazione degli integrali di equazioni lineari ellittico-paraboliche alle derivate parziali del secondo ordine.* Accad. Naz. dei Lincei. Atti Rendiconti (6) 5 (1927) pp. 138-143.

[17] M. PICONE: *Maggiorazione degli integrali delle equazioni totalmente paraboliche alle derivate parziali del secondo ordine.* Annali di Mat. Pura ed Applic. (4) 7 (1929) pp. 145-192.

[18] M. PICONE: *Sul problema della propagazione del calore in un mezzo privo di frontiera, conduttore, isotropo, e omogeneo.* Math. Ann. 101 (1929) pp. 701-712.

[19] M. H. PROTTER - H. F. WEINBERGER: *Maximum Principles in Differential Equations.* Prentice-Hall, Englewood Cliffs, N. J., 1967.

[20] D. H. SATTINGER: *A nonlinear parabolic system in the theory of combustion.* Quarterly of Applied Math. (in print).

[21] J. SCHRÖDER: *Invers-monotone Operatoren.* Arch. for Rat. Mech. and Anal. 10 (1962) pp. 276-295.

[22] J. SCHRÖDER: *Monotonie-Eigenschaften bei Differentialgleichungen.* Arch. for Rat. Mech. and Anal. 14 (1963) pp. 38-60.

[23] T. STYS: *On the unique solvability of the first Fourier problem for a parabolic system of linear differential equations of second order.* Prace Mat. 9 (1965) pp. 283-289.

[24] J. SZARSKI: *Sur la limitation et l'unicité des solutions d'un système non-linéaire d'équations paraboliques aux dérivées partielles du second ordre.* Ann. Polon. Math. 2 (1955) pp. 237-249.

[25] J. SZARSKI: *Sur un système non-linéaire d'inégalités différentielles paraboliques.* Ann. Polon. Math. 15 (1964) pp. 15-22.

[26] J. SZARSKI: *Differential Inequalities.* Polish Scientific Publishers, Warsaw, 1965.

[27] P. SZEPTYCKI: *Existence theorem for the first boundary value problem for a quasilinear elliptic system.* Acad. Polon. des Sciences, Bulletin 7 (1959) pp. 419-424.

Pervenuto in Redazione il 16 novembre 1974.

Tipografia " Oderisi „ Editrice Gubbio 1975