THE USE OF GEOMETRIC TOOLS IN THE BOUNDARY CONTROL OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: There has been new interest in the successful application of differential geometric methods in the control of p.d.e.'s. (See for example Contemporary Mathematics #268, AMS 2000, particularly the article by the present authors). Here we describe those results, and some newer results using the methods of curvature flows. We also present an example for which control is possible but cannot be proved by means of any convex function.

Although the subject of boundary control of partial differential equations is about a quarter of a century old, and that of Riemannian geometry much older still, until recently there has been relatively little interaction between the two. This is especially surprising in view of the rôle bicharacteristics play in boundary control, which naturally bring to mind geodesics— a basic concept in Riemannian geometry. We will describe some recently established links between the two subjects.

Our focus in this paper is on the qualitative relationship between Riemannian geometry and boundary control. Thus we shall not attempt here to express controllability in terms of the optimal choice of Sobolev spaces, leaving such questions to other papers such as [7]; nor shall we attempt to find the optimal smoothness of the Riemannian metric and of other coefficients of the hyperbolic equation.

Consider a compact, n-dimensional Riemannian manifold-with-boundary $\overline{\Omega}$. We assume that $\partial\Omega$ is smooth and nonempty, and that the metric of $\overline{\Omega}$ is smooth, *i.e.,* C^∞. We are interested in the boundary control of the following natural hyperbolic partial differential equation (Riemannian wave equation) on $\Omega \times [0, T]$:

$$\frac{\partial^2 u}{\partial t^2} = \Delta_g u := \sum_{i,j=1}^{n} \frac{1}{\gamma} \frac{\partial}{\partial x_i} \left(\gamma \, g^{ij}(x) \frac{\partial u}{\partial x_j} \right) \tag{1}$$

for all $(x, t) \in \Omega \times [0, T]$, where $(x_1, \ldots, x_n)$ are arbitrary local coordinates, $g^{ij}(x)$ are the entries of the inverse matrix to the coefficients $g_{ij}(x)$ of the Riemannian metric, and γ is the Riemannian volume integrand: $\gamma(x) = \sqrt{\det(g_{ij}(x))}$. We consider the problem of the control in time T of equation (1) from the boundary $\partial\Omega$. More precisely, we consider the boundary conditions

$$u(x,t) = U(x,t) \quad \text{for all} \quad (x,t) \quad \text{on} \quad \partial\Omega \times [0,T], \tag{2}$$

where $U \in H^{1/2}(\partial\Omega \times [0, T])$ is the control, *i.e.* a function which may be chosen as needed. The controllability question is whether, given any initial conditions

$$u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x), \tag{3}$$

with finite energy, there is a choice of controls $U \in H^{1/2}(\partial\Omega \times [0, T])$ such that the solution of (1) with initial conditions (3) and boundary conditions (2) vanishes identically on $\Omega \times [T, \infty)$. Equivalently, we ask whether for some choice of controls U the terminal Cauchy values vanish: $u(x, T) = 0$, $\frac{\partial u}{\partial t}(x, T) = 0$ for all x in Ω.

In this report we wish to describe two main results (theorems 1 and 2). We shall sketch proofs, referring to [1], [6] for complete proofs.

We define a chord to be a curve in $\overline{\Omega}$ of shortest length between two boundary points.

Theorem 1. Suppose that any two boundary points of the manifold $\overline{\Omega}$ are connected by a unique chord, which is nondegenerate. Assume that $\partial\Omega$ has positive second fundamental form. Then the hyperbolic equation (1) is controllable from $\partial\Omega$ by means of boundary conditions (2) in any time

$$T > T_0 := \mathrm{diam}_{\overline{\Omega}}(\partial\Omega).$$

Here, the *diameter* of the boundary of Ω is the maximum distance between any two of its points, with respect to the distance measured in $\overline{\Omega}$, that is: the length of the longest chord of $\overline{\Omega}$. Our convention for the sign of

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the second fundamental form is such that if Ω is a ball of radius r in $\mathbb{R}^n$, with the Euclidean metric, then $\partial\Omega$ has positive second fundamental form $B = r^{-1}ds^2$.

The "uniqueness" of a chord $\gamma : [0, a] \to \overline{\Omega}$ is understood *modulo* reparameterizations $s \mapsto \gamma(As + B)$ $(A, B \in \mathbb{R})$ of the independent variable s.

We shall refer to the infimal value T_0 as the "optimal time of control," even though T_0 itself may not be a control time.

The hypotheses of chord uniqueness and nondegeneracy may be viewed as follows. Consider any two boundary points p and q. Suppose that, among all the light rays leaving q simultaneously and propagating in Ω (without reflection), exactly one ray reaches p first. This is equivalent to the uniqueness of chords. Nondegeneracy of chords is equivalent to the statement that, in the situation just described, when a ray leaving q turns out to be a chord from q to p, the boundary point p depends in a diffeomorphic way on the initial direction of the ray at q.

As the reader will verify immediately, Theorem 1 follows from Propositions 1, 2 and 3 below.

For the propositions below, we let $\overline{\Omega}$ be extended to become a subset of an open n-dimensional Riemannian manifold M.

Proposition 1. *Let M be a Riemannian manifold. Then the bicharacteristics of equation (1) are the graphs in $M \times \mathbb{R}$ of geodesics of M, with unit-speed parameter identified with time $\in \mathbb{R}$.* See [4], p. 209.

Proposition 2. *If every bicharacteristic in $\overline{\Omega} \times (0, T)$ enters or leaves $\overline{\Omega} \times (0, T)$ across the lateral boundary $\partial\Omega \times (0, T)$, then boundary control is available in any time $\geq T$. Conversely, if there is a single bicharacteristic in $\Omega \times (0, T)$ that enters $\overline{\Omega} \times [0, T]$ through the open bottom $\Omega \times \{0\}$ and leaves through $\Omega \times \{T\}$, without hitting the lateral boundary, then boundary control in time T is not possible.*

Proof. See for ex. [8] where the proof which is given for bounded domains in $\mathbb{R}^n$ carries over without difficulties to manifolds (this is because it involves only interior propagation of singularity results). The assumption of real analytic coefficients is easily removed for time independent coefficients. The proof in [2] given for $\mathbb{R}^n$ holds with optimal Sobolev spaces. For converse see [9]. □

Proposition 3. *Assume that the boundary $\partial\Omega$ of the compact Riemannian manifold-with-boundary $\overline{\Omega}$ has positive second fundamental form. Suppose that any two points of $\partial\Omega$ are connected by a unique chord, which is nondegenerate. Then any interior geodesic segment $\gamma : (b-\varepsilon, b+\varepsilon) \to \Omega$ may be extended to a geodesic $\gamma : [s_0, s_1] \to \overline{\Omega}$ which is a chord, that is, which realizes the minimum length between two distinct points $\gamma(s_0)$ and $\gamma(s_1)$ in $\partial\Omega$.*

The proof uses an "open and closed" connectedness argument in the unit tangent bundle of $\overline{\Omega}$.

Theorem 2. If $m = 2, \partial\Omega$ is convex and there are no closed geodesics, then there is a finite time of control T_0.

Proposition 4. *(See [7]) suppose there is a C^2 function convex on $\overline{\Omega}$ with respect to the Riemannian metric g:*

$$Hess_g(X, X) \geq 2C|X|_g^2 \quad \forall\ X \in T_x(\Omega)\ ,$$

$$C > 0 \text{ and } 0 \leq v \leq K.$$

Then $T_0 \leq 2\sqrt{K/C}$. (See also [10])

Proposition 5. *(See also [3], [5].) Let F be a compact smooth Riemann surface with boundary. Let $C(0) : S^1 \to F$ be a smooth curve embedded in F. Then $C(t) : S^1 \to F$ exists for $t\epsilon[0, t_\infty)$ satisfying*

$$\frac{\partial C}{\partial t} = kN$$

where k is the curvature of C and N is a unit normal vector. Here $t_\infty \epsilon(O, \infty]$ is maximal. If t_∞ is finite then C converges to a point. If $t_\infty = \infty$ then a subsequence of $C(t)$ converges to a geodesic.

Thus if there is no geodesic, $C(t)$ converges to a point. Now if ∂F in convex and $C(0) = \partial F$, then after computing the parabolic PDE satisfied by the geodesic curvature of $C(t)$, it follows from the strong maximum principle of linear parabolic equations that all $C(t)$ are convex, and that distinct curves do not intersect. By reparametrizing the curve appropriately it is possible to build up a convex function having the curves $C(t)$ as level sets. This convex function will satisfy the condition of Proposition 4.

There is a sharpening of Proposition 4 (see [7]) which allows boundary control provided zero data is imposed on an inner "non controlled" part of the boundary - where the g-normal derivative of v is required to be non-positive. This allows control even in the case of the existence of a geodesic - provided on an "inner boundary", close to the geodesic, zero data is imposed.

Since the work of [2], it has been assumed in specialized PDE control circles that, even in the case of the Euclidean Laplacian, a 'minimal' controlled portion of the boundary—such as is provided by the sharp geometric optics condition [2]—may not be expected, in general, to be obtained by using classical multipliers with vector fields which are coercive, in particular the gradients of strictly convex functions. Here we provide a specific example with a full analysis in the present context.

We construct an example of a domain in $\mathbb{R}^2$ with controls on the outer of the four boundary components and $g_{ij}(x) \equiv \delta_{ij}$, therefore with the Euclidean Laplacian $\Delta_g = \Delta$, so that control is achieved in a finite time T_0, but there is no strictly convex function v on Ω with the additional property of nonpositive normal derivative on the uncontrolled boundary (see Figure 1.) In fact, the analysis of [2], [8] shows that control may be achieved in a time equal to the maximum length of geodesics which reflect with equal angles at the uncontrolled boundary, before they cross the controlled boundary. In Figure 1, this will be achieved by a polygonal curve consisting of two line segments, each of which grazes the outer boundary curve, reflecting with equal angles at the upper boundary circle. On the other hand, a strictly convex function v on Ω must have positive outward normal derivative somewhere on one of the three uncontrolled boundary circles. Namely, let the black dot in Figure 1 indicate the origin. The segments l_1, l_2 and $l_3 \subset \Omega$ of the rays from the origin through the centers of the three circles meet the uncontrolled boundary at right angles at x_1, x_2 and x_3, resp. If the outward normal derivative of v at x_k is ≤ 0, then since the restriction of v to l_k is strictly convex, the derivative of v at the origin in the x_k direction is strictly negative. But x_1, x_2 and x_3 do not lie in any half-plane of $\mathbb{R}^2$, so this contradicts the differentiability of v at the origin.

On the other hand, the Euclidean Laplacian in $\mathbb{R}^n$ always admits infinitely many strictly convex functions, e.g., the quadratic functions $v(x) = \|x - x_0\|^2$, where x_0 is *any* fixed point in $\mathbb{R}^n$. Thus, by Proposition 4 strengthened as quoted above from [7], the above example in $\mathbb{R}^2$ with the Euclidean Laplacian is always exactly controllable in optimal Sobolev spaces over a time $T >= T_0$, provided that we apply control on an additional portion of the boundary. Namely, we have to retain control on the entire portion of the boundary where $\nabla v \cdot n \geq 0$, n being the unit outward normal. One symmetric way to do this is as follows. Call now x_0 the center of the domain (the dot in the picture). Then, apply the aforementioned strengthened version of Proposition 4 with the strictly convex function $v(x) = \|x - x_0\|^2$. This requires that we apply control not only on the exterior boundary, as done before, but also on the arcs of each circle illuminated by a light source at x_0.

An interesting 2-d example (with the Euclidean Laplacian) is given in [2, p. 1031, Fig. 4]. It displays a disconnected 'minimal' portion of a circumference, which is sufficient for control in light of the geometric optics criterion. As there is no discussion, however, on whether or not such example could also be obtained by a strictly convex function and the aforementioned strengthened version of Proposition 4, we warn the reader—who may be induced to the opposite conclusion by the last paragraph on p. 1031 of [2]—that an argument similar to the one provided for our own example of our Figure 1 shows likewise that Fig. 4 of [2] cannot be obtained by a strictly convex function.

We finally notice that a non-Euclidean example has been constructed by Michael Galbraith in which control on the whole boundary in finite time is possible, but there is no strictly convex function whatever.

Figure 1: The outer boundary is subject to Dirichlet controls. The three inner boundary circles are uncontrolled; instead, homogeneous Dirichlet boundary conditions are imposed. No line segment inside the domain joins distinct inner boundary circles, which implies finite-time control. But the three inner boundary circles may be reached from the point at center, in directions not lying in a half space; this implies that there is no convex function which would ensure finite-time control.

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