SIXTH PROBLEM SET
Math 5615H: Honors Analysis

Due W 18 October, 2017.
12 points each; total 60 points.

1. If $a_1 = 1$, and
$$a_{n+1} = \frac{1}{1+a_n} \text{ for } n = 1, 2, \ldots,$$
prove that the sequence $\{a_n\}$ converges and find its limit. You can use without proof the fact that
$$\lim_{n\to\infty} q^n = 0 \text{ if } |q| < 1.$$

2. Let X be the metric space of bounded sequences of real numbers: $x = \{x_1, x_2, \ldots, x_n, \ldots\}$ with the distance
$$d(x, y) := \sup_n |x_n - y_n| \text{ for } x = \{x_n\},\ y = \{y_n\}.$$
Let c_n be a sequence of positive numbers. Show that the set
$$K := \{x = \{x_n\} \in X : \quad |x_n| \le c_n \text{ for all } n = 1, 2, \ldots\}$$
is compact in X if and only if $c_n \to 0$ as $n \to \infty$. (You may assume, but do not have to show, that X is a metric space.)

3. Let a, b be arbitrary real numbers, set $x_1 = a$ and $x_2 = b$ and let
$$x_n := \frac{1}{2}(x_{n-2} + x_{n-1}) \text{ for } n = 3, 4, 5, \ldots.$$
Show that the sequence $\{x_n\}$ converges, and find its limit.

4. Exercise 30 on p.46 of Rudin.

5. Exercise 8 on p.78 of Rudin.