Math 2263 Spring 2009 Midterm 3, WITH SOLUTIONS April 23, 2009

1. (10 points) Let B be the box, or rectangular solid: $0 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 1$. Find

$$\iiint_B (xy - 2yz + x^2z^2)\,dV.$$

SOLUTION:

$$\iiint_B (xy - 2yz + x^2z^2)\,dV = \int_0^1 \int_0^3 \left[\frac{x^2y}{2} - 2yzx - \frac{x^3z^2}{3}\right]_{x=0}^{2} dy\,dz =$$

$$\int_0^1 \int_0^3 \left(2y - 4yz + \frac{8z^2}{3}\right) dy\,dz = \int_0^1 \left[y^2 - 2y^2z + \frac{8z^2y}{3}\right]_{y=0}^{3} dz = \int_0^1 (9 - 18z + 8z^2)\,dz = 9 - 9 + \frac{8}{3} = \frac{8}{3}.$$

2. (15 points) Let E be the solid region bounded below by the cone $z^2 = x^2 + y^2$ and above by the plane $z = 1$. Find

$$\iiint_E (x^2 + y^2)\,dV.$$

Hint: Try cylindrical coordinates.

SOLUTION: Use cylindrical coordinates: E is described by $0 \le r \le z \le 1$, $0 \le \theta \le 2\pi$. So

$$\iiint_E (x^2 + y^2)\,dV = \int_0^1 \int_0^{2\pi} \int_0^z r^2\, r\,dr\,d\theta\,dz = 2\pi \int_0^1 \frac{z^4}{4}\,dz = \frac{\pi}{10}.$$

3. (25 points) C_1 and C_2 are oriented curves in the (x,y)-plane, each of which starts at $(0,0)$ and ends at $(1,1)$. C_1 is given by $y = x^2$, $0 \le x \le 1$; and C_2 is given by $x = y^2$, $0 \le y \le 1$. Let the vector field $\vec{F}$ be given by $\vec{F}(x,y) = 2xy\,\vec{\imath} + (x^2 - y^2)\,\vec{\jmath}$.

(a) (10 points) Find $\int_{C_1} \vec{F} \cdot d\vec{r}$.

SOLUTION:

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \Big(2xy\,dx + (x^2 - y^2)\,dy\Big) = \int_0^1 [2x(x^2) + (x^2 - x^4)2x]\,dx = \left[4\frac{x^4}{4} - 2\frac{x^6}{6}\right]_0^1 = \frac{2}{3}.$$

(b) (10 points) Find $\int_{C_2} \vec{F} \cdot d\vec{r}$.

SOLUTION:

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 [2(y^2)y(2y) + ((y^2)^2 - y^2)]\,dy = \int_0^1 [4y^5 + y^4 - y^2]\,dy = \left[4\frac{y^6}{6} + \frac{y^5}{5} - \frac{y^3}{3}\right]_{y=0}^{1} = \frac{8}{15}.$$

(c) (5 points) Is the vector field $\vec{F}(x,y)$ **conservative**? Why or why not?

SOLUTION: NO! $\frac{2}{3} \ne \frac{8}{15}$, so the integral from $(0,0)$ to $(1,1)$ depends on the path of integration connecting the two points.

4. (15 points) Under the transformation $x = 4u + v$, $y = 5u + 2v$ from the (u, v)-plane to the (x, y)-plane, the circular disk D given by the inequality $u^2 + v^2 \leq 9$ is transformed into the elliptical region E given by
$$85\, x^2 - 76\, xy + 17\, y^2 \leq 9.$$
Compute the **area of E as an integral over D.**

SOLUTION: First compute the Jacobian determinant:
$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = (4)(2) - (1)(5) \equiv 3.$$
Also, the area of a circular disk of radius 3 is $A(D) = \pi(3)^2 = 9\pi$. So
$$A(E) = \iint_D \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA_{u,v} = \iint_D 3\, dA_{u,v} = 27\pi.$$

5. (20 points) Let C be the circle $x^2 + y^2 = 9$ in the (x, y)-plane, oriented counterclockwise. Find
$$\oint_C (2xye^{x^2} + 2y^2 - y)\, dx + (e^{x^2} + 4xy - 3x)\, dy.$$
Hint: try Green's theorem!

SOLUTION: Write $P(x, y) = 2xye^{x^2} + 2y^2 - y$ and $Q(x, y) = e^{x^2} + 4xy - 3x$. Following the hint, C is the oriented boundary of the circular disk D: $x^2 + y^2 \leq 9$, so by Green's Theorem $\oint_C P\, dx + Q\, dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$. But $\frac{\partial Q}{\partial x} = 2xe^{x^2} + 4y - 3$ and $\frac{\partial P}{\partial y} = 2xe^{x^2} + 4y - 1$. Subtracting, you get $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \equiv -2$ for all (x, y), so
$$\oint_C P\, dx + Q\, dy = \iint_D (-2)\, dA = -18\pi.$$

6. (15 points) The portion E of the ball of radius 2 with center at $(0, 0, 0)$, which lies above the cone
$$z = \sqrt{x^2 + y^2},$$
is described in spherical coordinates by $0 \leq \rho \leq 2$, $0 \leq \phi \leq \pi/4$, $0 \leq \theta \leq 2\pi$. Find the **volume** of this figure by computing an integral in spherical coordinates. (*Hint:* Recall that $\sin \pi/4 = \frac{1}{2}\sqrt{2} = \cos \pi/4$.)

SOLUTION: Using spherical coordinates, the volume of E is
$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = 2\pi \int_0^{\pi/4} \left[\frac{\rho^3}{3}\right]_{\rho=0}^{2} \sin\phi\, d\phi =$$
$$2\pi \frac{8}{3}\Big[-\cos\phi\Big]_0^{\pi/4} = \frac{8\pi}{3}(2 - \sqrt{2}).$$