Math 2263 Spring 2009 Midterm 1, WITH SOLUTIONS February 19, 2009

1. (15 points) Find an equation for the **plane** passing through the two points $\langle x, y, z\rangle = \langle 4, -2, 2\rangle$ and $\langle 1, 0, -2\rangle$ so that the vector $\vec{i} + \vec{j} + \vec{k}$ is tangent to the plane.

   SOLUTION: A normal vector $\vec{v}$ is the cross product of the vector $\langle 3, -2, 4\rangle$ from one given point to the other and $\langle 1, 1, 1\rangle$.

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 4 \\ 1 & 1 & 1 \end{vmatrix} = -6\vec{i} + \vec{j} + 5\vec{k}.$$

   Since $\langle 1, 0, -2\rangle$ is in the plane, an equation for the plane is $-6(x-1) + (y-0) + 5(z+2) = 0$, or simplifying:

$$-6x + y + 5z = -16.$$

2. (15 points) Find an equation for the surface in (x, y, z)-space obtained by **rotating** the ellipse $x^2 + 4y^2 = 1$ of the (x, y)-plane **about the x-axis.**

   SOLUTION: The distance from the x-axis is $\sqrt{y^2 + z^2}$, which replaces $|y|$ in the given equation. So the equation for the surface of revolution is $x^2 + 4y^2 + 4z^2 = 1$.

3. (15 points) The lines given parametrically by

$$(x, y, z) = (7 + 2t, -1 - t, -2t), \quad -\infty < t < \infty$$

   and

$$(x, y, z) = (4 - s, -1 + 2s, 2 + 2s), \quad -\infty < s < \infty$$

   intersect at the point $\langle x, y, z\rangle = \langle 3, 1, 4\rangle$. Find an equation for the **plane** which contains both lines.

   SOLUTION: A vector in the direction of the first line is $\vec{u} = 2\vec{i} - \vec{j} - 2\vec{k}$, and for the second line $\vec{w} = -\vec{i} + 2\vec{j} + 2\vec{k}$. So a normal vector $\vec{v}$ to the plane is the cross product:

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 2\vec{i} - 2\vec{j} + 3\vec{k}.$$

   An equation for the plane is $2(x-3) - 2(y-1) + 3(z-4) = 0$, or simplifying:

$$2x - 2y + 3z = 16.$$

4. (15 points) For the function $f(x, y) = e^{2x+y}\sin y$, find the **second partial derivatives**

$$f_{xx}, \quad f_{xy} \quad \text{and} \quad f_{yy}.$$

   SOLUTION: Compute $f_x = 2e^{2x+y}\sin y$ and $f_y = e^{2x+y}[\sin y + \cos y]$. Then

$$f_{xx} = 4e^{2x+y}\sin y, \quad f_{xy} = 2e^{2x+y}[\sin y + \cos y]$$

$$\text{and} \quad f_{yy} = e^{2x+y}[\sin y + \cos y + \cos y - \sin y] = 2e^{2x+y}\cos y.$$

5. (10 points) Suppose $z = f(x,y)$ is a function with first partial derivatives $f_x(1,2) = 3$ and $f_y(1,2) = 5$. If x and y are both functions of t: $x = g(t) = -1 - 2t$ and $y = h(t) = 6 + 4t$, find the **derivative at** $t = -1$:

$$\frac{dz}{dt} = \frac{d}{dt} f\Big(g(t), h(t)\Big).$$

SOLUTION: The **chain rule** says that

$$\frac{dz}{dt} = f_x\Big(g(t), h(t)\Big)\frac{dx}{dt} + f_y\Big(g(t), h(t)\Big)\frac{dy}{dt}.$$

Compute that when $t = -1$, $x(-1) = 1$ and $y(-1) = 2$, so in the chain rule, both f_x and f_y are evaluated at $(1,2)$. We get:

$$\frac{dz}{dt}(-1) = (3)(-2) + (5)(4) = 14.$$

6. (15 points) The point $\langle x,y,z\rangle = \langle 1,-1,2\rangle$ lies on the surface S:

$$x^2 + z^2 - 2xy - y^2 = 6.$$

Find the equation of the **tangent plane** to the surface S at $\langle 1,-1,2\rangle$. Write it in the form $ax + by + cz = d$.

SOLUTION: Compute the **gradient** of $g(x,y,z) = x^2 + z^2 - 2xy - y^2$: $\vec{\nabla} g = (2x - 2y)\vec{i} + (-2x - 2y)\vec{j} + 2z\vec{k}$. Then $\vec{\nabla} g(1,-1,2) = 4\vec{i} + 4\vec{k}$ is a normal vector to the surface S given by $g(x,y,z) = 6$ at $\langle 1,-1,2\rangle$. The equation of the **tangent plane** to S at $\langle 1,-1,2\rangle$ is

$$\vec{\nabla} g \cdot \Big(\langle x,y,z\rangle - \langle 1,-1,2\rangle\Big) = 4(x-1) - 0(y+1) + 4(z-2) = 0, \quad \text{or} \quad x + z = 3.$$

7. **(15 points) (a)**Find the **gradient** of the function $f(x,y,z) = e^x \ln(xy + z^2)$ at the point $\langle x,y,z\rangle = \langle -1,1,2\rangle$.

(15 points) **SOLUTION:** $f_x = e^x \ln(xy + z^2) + \frac{ye^x}{xy+z^2}$; $f_y = \frac{xe^x}{xy+z^2}$; and $f_z = \frac{2ze^x}{xy+z^2}$. So the partial derivatives of f at $\langle x,y,z\rangle = \langle -1,1,2\rangle$ are $f_x = \frac{\ln 3}{e} + \frac{1}{3e}$, $f_y = -\frac{1}{3e}$ and $f_z = \frac{4}{3e}$. The gradient of f is

$$\vec{\nabla} f(-1,1,2) = \Big[\frac{\ln 3}{e} + \frac{1}{3e}\Big]\vec{i} - \frac{1}{3e}\vec{j} + \frac{4}{3e}\vec{k}.$$

(b) Find the **directional derivative** of the function $f(x,y,z) = e^x \ln(xy + z^2)$ at the point $\langle x,y,z\rangle = \langle -1,1,2\rangle$ in the direction of the unit vector

$$\vec{u} = \frac{1}{3}\Big(2\vec{i} - 2\vec{j} + \vec{k}\Big).$$

SOLUTION: Since $\vec{u}$ is a unit vector, we know that $D_{\vec{u}} f(-1,1,2) = \vec{u} \cdot \vec{\nabla} f(-1,1,2) = \frac{1}{3}\Big(2\vec{i} - 2\vec{j} + \vec{k}\Big) \cdot \Big([\frac{\ln 3}{e} + \frac{1}{3e}]\vec{i} - \frac{1}{3e}\vec{j} + \frac{4}{3e}\vec{k}\Big) = \frac{1}{3}\Big(2[\frac{\ln 3}{e} + \frac{1}{3e}] - 2\frac{1}{3e} + \frac{4}{3e}\Big) = \frac{2}{3e}\ln 3 + \frac{4}{9e}$.