Math 2263 Fall 2008 Midterm 3, WITH SOLUTIONS November 25, 2008

1. (15 points) Let B be the box, or rectangular solid: $0 \le x \le 2$, $0 \le y \le 1$, $0 \le z \le 3$. Find

$$\iiint_B (x^2 + y)(x - z^2)\, dV.$$

SOLUTION: $\iiint_B (x^2 + y)(x - z^2)\, dV = \iiint_B (x^3 + xy - x^2z^2 - yz^2)\, dV =$
$\int_0^3 \int_0^1 \left[\frac{x^4}{4} + \frac{yx^2}{2} - \frac{x^3z^2}{3} - xyz^2\right]_{x=0}^2 dy\, dz = \int_0^3 \int_0^1 \left[4 + 2y - \frac{8}{3}z^2 - 2yz^2\right] dy\, dz = \int_0^3 (5 - \frac{11}{3}z^2)\, dz =$
$\left[5z - \frac{11}{9}z^3\right]_{z=0}^3 = -18.$

2. (15 points) Let E be the cylindrical solid $x^2 + y^2 \le 9$, $0 \le z \le 1$. Find

$$\iiint_E z\, e^{x^2+y^2}\, dV.$$

SOLUTION: Use cylindrical coordinates: $0 \le \theta \le 2\pi$, $0 \le r \le 3$, $0 \le z \le 1$. Then
$\iiint_E z\, e^{x^2+y^2}\, dV = \int_0^1 \int_0^{2\pi} \int_0^3 z e^{r^2} r\, dr\, d\theta\, dz = \int_0^1 z\, dz \int_0^{2\pi} d\theta \int_0^9 e^u \frac{1}{2}\, du = \frac{\pi}{2}\left(e^9 - 1\right).$

3. (20 points) The set E in $\mathbb{R}^3$ is described by the inequalities:

$$\sqrt{x^2 + y^2} \le z \le \sqrt{9 - x^2 - y^2}.$$

(a) (5 points) Write a description of E in **spherical coordinates**.

SOLUTION: E is described by $\rho^2 = x^2+y^2+z^2 \le 9$ and $x^2+y^2 = \rho^2 \sin^2\phi \le z^2 = \rho^2\cos^2\phi$, so

$$0 \le \rho \le 3 \quad \textit{and} \quad 0 \le \phi \le \frac{\pi}{4}.$$

(b) (15 points) Compute the **volume** of E.

SOLUTION: The volume integrand in spherical coordinates is $dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$. So the volume of E is

$$\int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^3 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = 2\pi \int_0^{\frac{\pi}{4}} \sin\phi\, d\phi \int_0^3 \rho^2\, d\rho = 9\pi(2 - \sqrt{2}).$$

4. (25 points) Let C be the circle $x^2 + y^2 = 100$ in the (x, y)-plane, oriented counterclockwise. Find $\oint_C (2xy + e^x)\, dx + (x^2 - \sin y + 3x)\, dy$. *Hint:* try Green's theorem.

SOLUTION: Write $P(x, y) = 2xy + e^x$ and $Q(x, y) = x^2 - \sin y + 3x$. Following the hint, we compute the other side of the formula for Green's Theorem:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (2x + 3) - (2x) = 3.$$

The circle C, travelling counterclockwise, is the positively oriented boundary of the disk D described by $x^2 + y^2 \leq 100$. By Green's Theorem,

$$\oint_C (P\,dx + Q\,dy) = \iint_D 3\,dA = 300\pi,$$

the area of D times 3.

5. (25 points) C_1 and C_2 are oriented curves in the (x, y)-plane, each of which starts at $(0, 0)$ and ends at $(2, 2)$. C_1 is given by the parameterization $\vec{r}(t) = t\vec{i} + (3t - t^2)\vec{j}$, $0 \leq t \leq 2$; and C_2 is given by the parameterization $\vec{r}(t) = t\vec{i} + (t^2 - t)\vec{j}$, $0 \leq t \leq 2$.

(a) (10 points) Find $\int_{C_1} xy\,dx + (y - 3x)\,dy$.

SOLUTION: $\int_{C_1} xy\,dx + (y-3x)\,dy = \int_0^2 [t(3t-t^2) - (3t-t^2-3t)(3-2t)]\,dt = \int_0^2 (3t^2 - t^3 - 3t^2 + 2t^3)\,dt = \left[\frac{t^4}{4}\right]_0^2 = 4$.

(b) (10 points) Find $\int_{C_2} xy\,dx + (y - 3x)\,dy$.

SOLUTION: $\int_{C_2} xy\,dx + (y-3x)\,dy = \int_0^2 [t(t^2-t) + (t^2-4t)(2t-1)]\,dt = \left[\frac{3t^4}{4} - \frac{10}{3}t^3 + 2t^2\right]_0^2 = -\frac{20}{3}$.

(c) (5 points) Is the vector field $\vec{F}(x, y) = xy\vec{i} + (y - 3x)\vec{j}$ **conservative**? Why or why not?

SOLUTION: $\vec{F}$ is **not** a conservative vector field, because the integral along a curve from $(0, 0)$ to $(2, 2)$ of $\vec{F} \cdot d\vec{r}$ depends on the curve: for C_1 it equals 4, and for C_2 it equals $-\frac{20}{3}$. In fact, if $\vec{F}$ **were** conservative, then both would equal $f(2, 2) - f(0, 0)$ where $\vec{F} = \vec{\nabla} f$.